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GROWTH OF SOBOLEV NORMS IN QUASI-INTEGRABLE QUANTUM SYSTEMS

BY DARIO BAMBUSI AND BEATRICE LANGELLA

ABSTRACT. – We prove an abstract result giving a $\langle t\rangle^{\varepsilon}$ upper bound on the growth of the Sobolev norms of a time-dependent Schrödinger equation of the form $\mathrm{i}\dot{\psi} = H_0\psi + V(t)\psi$. Here H_0 is assumed to be the Hamiltonian of a steep quantum integrable system and to be a pseudodifferential operator of order $\mathtt{d} > 1$; $V(t)$ is a time-dependent family of pseudodifferential operators, unbounded, but of order $\mathtt{b} < \mathtt{d}$. The abstract theorem is then applied to perturbations of the quantum anharmonic oscillators in dimension 2 and to perturbations of the Laplacian on a manifold with integrable geodesic flow, and in particular Zoll manifolds, rotation-invariant surfaces and Lie groups. The proof is based on a quantum version of the proof of the classical Nekhoroshev theorem.

RÉSUMÉ. – Nous prouvons un résultat abstrait donnant une majoration de la forme $\langle t\rangle^{\varepsilon}$ pour la croissance des normes de Sobolev d'une équation de Schrödinger dépendant du temps de la forme $\mathrm{i}\dot{\psi} = H_0\psi + V(t)\psi$. On suppose que H_0 est l'hamiltonien d'un système quantique intégrable escarpé (steep) et qu'il est un opérateur pseudo-différentiel d'ordre $\mathtt{d} > 1$; $V(t)$ est une famille dépendant du temps d'opérateurs pseudo-différentiels, non bornés, mais d'ordre $\mathtt{b} < \mathtt{d}$. Le théorème abstrait est ensuite appliqué aux perturbations des oscillateurs quantiques anharmoniques en dimension 2 et aux perturbations du laplacien sur une variété avec flot géodésique intégrable, et en particulier sur les variétés de Zoll, les surfaces invariantes par rotation et les groupes de Lie. La démonstration repose sur une version quantique de la preuve du théorème de Nekhoroshev classique.

1. Introduction

In this paper we prove an abstract theorem giving a $\langle t\rangle^{\varepsilon}$ upper bound for the Sobolev norms of the solutions of an abstract Schrödinger equation of the form

$$\mathrm{i}\dot{\psi} = (H_0 + V(t))\psi, \quad \psi \in \mathcal{H}, \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space, and H_0 is the Hamiltonian of a quantum system which is globally integrable in a sense defined below and steep; H_0 is also assumed to be a pseudodifferential operator of order $\mathtt{d} > 1$, and $V(t)$ is a smooth time-dependent family of self-adjoint pseudodifferential operators of order $\mathtt{b} < \mathtt{d}$. In this sense V can be considered a perturbation of H_0 and the system $H_0 + V(t)$ can be called *quasi-integrable*.

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The novelty of this result is twofold: it extends the class of unperturbed systems for which upper bounds of the Sobolev norms can be obtained, and it allows to treat the case of unbounded perturbations. We emphasize that the class of unperturbed systems treated here strictly contains *all* the systems of order $\mathtt{d} > 1$ for which estimates on the growth of Sobolev norms have been obtained; it also contains new systems, like the quantum 2-d anharmonic oscillator and Lie groups.

We now describe more in detail our result. First, following [6] (see also [23]) we introduce some abstract algebras of linear operators in $\mathcal{H}$ that enjoy the properties typical of pseudodifferential operators. We actually call the elements of these algebras pseudodifferential operators.

Then we give the definition of globally integrable quantum system. The idea is to introduce some operators A_j, $j = 1, \dots, d$ that are the quantum analogue of the classical action variables and to consider quantum systems with Hamiltonian $H_0 = h_0(A_1, \dots, A_d)$, namely Hamiltonians which are functions of the quantum action variables. In turn, the quantum action variables are defined to be d commuting self-adjoint pseudodifferential operators of order 1, whose joint spectrum is contained in $\mathbb{Z}^d + \kappa$, with some $\kappa \in \mathbb{R}^d$. The motivation of this definition rests in the classical results by Duistermaat-Guillemin [21], Colin de Verdière [42], and Helffer-Robert [27], which ensure that, under suitable assumptions, the quantization of a classical action variable is a perturbation of a pseudodifferential operator with spectrum contained in $\mathbb{Z} + \kappa$ with $\kappa \in \mathbb{R}$. We point out that our definition, which does not involve directly the action variables of the classical system, is quite flexible, since it applies also to systems whose classical action variables are poorly known and to quantization of some superintegrable systems [34, 22] in which the Hamiltonian is independent of some of the action variables. Finally, we assume that the function h_0 is homogeneous of degree $\mathtt{d} > 1$ and fulfills the steepness assumption of the classical Nekhoroshev's theorem (see Definition 2.8 below). Concerning the perturbation V, we assume that it is a pseudodifferential operator of order $\mathtt{b} < \mathtt{d}$. In the case $\mathtt{b} = \mathtt{d}$, V cannot be considered as a perturbation of H_0. For this reason, we do not think it can be treated within our framework.

We come now to the applications of the abstract result. The first application is to perturbations of a 2-dimensional quantum anharmonic oscillator with Hamiltonian

$$H_0 = -\frac{\Delta}{2} + \frac{\|x\|^{2\ell}}{2\ell}, \quad x \in \mathbb{R}^2, \quad \ell \in \mathbb{N}, \quad \ell \geq 2, \tag{1.2}$$

for which we prove the $\langle t \rangle^{\varepsilon}$ upper bound on the growth of Sobolev norms. We recall that for anharmonic oscillators the situation was clear only for the 1-d case [6], while for the higher dimensional case only an estimate by $\langle t \rangle^{s}$ for the $\mathcal{H}^s$ norm was known for the case of bounded perturbations (see [32]), while, for unbounded perturbations, an upper bound of the form $\langle t \rangle^{s\alpha_s}$, with an exponent $\alpha_s > 1$ which diverges as $\mathtt{b} \to \mathtt{d}$, was proven in [32].(1) A $\langle t \rangle^{\varepsilon}$ estimate was out of reach for the 2d case with previous methods. The second application is to Schrödinger equations on compact manifolds with globally integrable geodesic flow, namely manifolds in which the Hamiltonian of the geodesic flow is integrable, and furthermore the action variables are globally defined and the Hamiltonian is steep. In this

(1) Actually the result of [32] applies to very general systems and does not rely on a pseudo-differential setting, but typically only allows to prove estimates with the exponent of $\langle t \rangle$ which depends on s.

paper, in order to be determined, we present some specific examples. First (i) we recover the known results for tori [17, 19, 14, 9] and Zoll Manifolds [6], then (ii) we consider rotation invariant surfaces (following [42, 19]) and construct the operators A_j by quantizing the classical action variables. The novelty of the result we get for rotation invariant surfaces is that we can deal with *unbounded* perturbations of the Laplacian. Another example (iii) is that of the Schrödinger equation on a Lie group (see [15, 13] for a KAM type result): it is known that the geodesic flow on a compact Lie group is integrable ([33, 16]), but very little is known on the action angle variables. For this reason we work directly at a quantum level. To this end we use the intrinsic pseudodifferential calculus on Lie groups developed in [23, 39] and construct directly the quantum actions in the case of compact, simply connected Lie groups. Here the lattice $\mathbb{Z}^d$ is essentially the lattice of the dominant weights of the irreducible representations of the Lie group. The result controlling growth of Sobolev norms for solutions of the Schrödinger equation on Lie groups is new.

We come now to a description of the proof of the abstract result (namely Theorem 2.10). The present paper is a direct continuation of the works [8, 10, 9, 11], and is based on the quantization of the proof of the classical Nekhoroshev theorem. Here in particular we develop an abstract version of the proof, which is based just on the use of the lattice of the joint spectrum of the actions.

Precisely, the proof consists of two steps: first one uses the methods of normal form to construct iteratively a family of unitary time-dependent operators conjugating the original Hamiltonian (1.1) to a Hamiltonian of the form

$$H_0 + Z(t) + R(t) \tag{1.3}$$

with $Z(t)$ which is a "normal form" operator (see Definition 4.4 below) and $R(t)$ a smoothing operator, which plays the role of a remainder. This was done in [10, 9] for the Schrödinger operator on $\mathbb{T}^d$ by quantizing the classical normal form procedure. We remark that the use of pseudo-differential calculus is what allows in [10, 9] to deal with unbounded perturbations. The second step of the proof in [10, 9] consists in analyzing the structure of the normal form operator, namely in studying the way it couples different Fourier modes depending on the resonance relations fulfilled by the frequencies of the classical system. This is essentially a quantum version of the geometric construction of Nekhoroshev theorem (see e.g., [35, 36, 26, 7] for the classical construction). As a result we obtain that $Z(t)$ has a block diagonal structure with dyadic blocks, so that for the dynamics of $H_0+Z(t)$ the Sobolev norms remain bounded forever. The addition of the remainder $R(t)$ is the responsible for the $\langle t\rangle^{\varepsilon}$ estimate on the growth.

To realize an abstract version of the above construction, one has to develop several new tools. A major difference with respect to the works [8, 10, 9, 11] is that we use here directly pseudo-differential operators, never dealing with the corresponding symbols. This enables to work only with the quantum actions, instead of the classical ones. In turn, the quantum actions are used to define a Fourier expansion of pseudo-differential operators, in which the Fourier coefficients are labeled by the points of the lattice of their joint spectrum. This is the main new technical ingredient of the present paper.

We point out that the geometric part of the proof is more complicated here than in [10, 9] because we deal here with the steep case, while in [10, 9] only the convex case was treated.