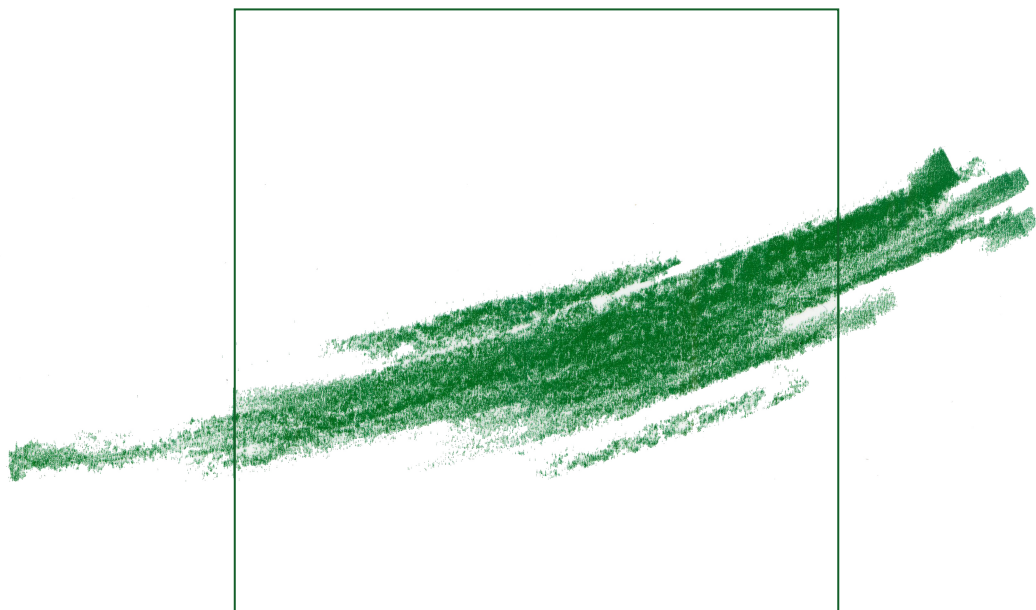


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# Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23



# **INTEGRAL GEOMETRY FROM BUFFON TO GEOMETERS OF TODAY**

**Rémi Langevin**

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**Rémi Langevin**

**Société Mathématique de France 2015**



# CONTENTS

|                                                                                 |    |
|---------------------------------------------------------------------------------|----|
| <b>Introduction</b>                                                             | 7  |
| <b>1. The beginning of integral geometry</b>                                    | 9  |
| 1.1. Buffon                                                                     | 9  |
| 1.2. The birth of the notion of geometric measure                               | 11 |
| <b>2. The Euclidean plane</b>                                                   | 15 |
| 2.1. Geometric measures on sets of lines                                        | 15 |
| 2.2. The Gauss map                                                              | 20 |
| <b>3. Two dimensional convex bodies and translations</b>                        | 27 |
| 3.1. Envelopes                                                                  | 27 |
| 3.2. Support functions and hérissos                                             | 29 |
| 3.3. Minkowski sum and mixed volumes                                            | 34 |
| <b>4. Blaschke's formulas and kinematic formulas</b>                            | 39 |
| 4.1. Poincaré's formulas                                                        | 39 |
| 4.2. Blaschke's formulas                                                        | 41 |
| <b>5. Grassmann manifolds and flag spaces</b>                                   | 45 |
| 5.1. Grassmann manifolds                                                        | 45 |
| 5.2. Invariant metrics and measures                                             | 48 |
| 5.3. Flags                                                                      | 49 |
| <b>6. Integral geometry of polyhedral surfaces in <math>\mathbb{R}^3</math></b> | 51 |
| 6.1. Theorema egregium                                                          | 52 |
| 6.2. Gauss-Bonnet theorem                                                       | 53 |
| 6.3. Projections on lines                                                       | 54 |
| 6.4. Global integral geometry of polyhedral surfaces                            | 55 |
| <b>7. Surfaces and curves in space</b>                                          | 57 |
| 7.1. Cauchy-Crofton formulas in $\mathbb{R}^3$                                  | 57 |
| 7.2. The Gauss map and the principal curvatures                                 | 58 |
| 7.3. Total curvature of curves in $\mathbb{R}^3$                                | 63 |
| <b>8. Integral geometry and topology</b>                                        | 67 |
| 8.1. Critical points and Gauss curvature, Chern and Lashof's theorem            | 67 |

|                                                                                             |            |
|---------------------------------------------------------------------------------------------|------------|
| 8.2. Total curvature of closed curves and knots .....                                       | 68         |
| 8.3. More theorems involving the topology of an immersion or an embedding .....             | 69         |
| <b>9. Tight immersions .....</b>                                                            | <b>75</b>  |
| 9.1. Definition .....                                                                       | 75         |
| 9.2. Plane curves and planar domains .....                                                  | 76         |
| 9.3. Surfaces .....                                                                         | 79         |
| <b>10. 3-dimensional convex bodies and related matters .....</b>                            | <b>85</b>  |
| 10.1. Support function .....                                                                | 85         |
| 10.2. Quermassintegrals and Steiner's formula .....                                         | 86         |
| 10.3. Orthogonal projections, polar varieties, and length of surfaces of $\mathbb{R}^3$ ... | 90         |
| 10.4. Tubes (2) .....                                                                       | 92         |
| 10.5. Localization of the length $L_1$ .....                                                | 97         |
| 10.6. Variation of a functional, linear kinematic formulas .....                            | 100        |
| <b>11. Integral geometry in spheres .....</b>                                               | <b>107</b> |
| 11.1. The spherical formula of Cauchy and Crofton .....                                     | 107        |
| 11.2. Flags .....                                                                           | 109        |
| <b>12. Integral geometry of foliations .....</b>                                            | <b>117</b> |
| 12.1. Codimension 1 foliations of a domain of $\mathbb{R}^3$ .....                          | 117        |
| 12.2. Foliations of surfaces of constant curvature .....                                    | 130        |
| 12.3. Codimension one foliations of spaces of constant curvature<br>in dimension 3 .....    | 142        |
| 12.4. Tight foliations .....                                                                | 143        |
| 12.5. Foliations of codimension higher than one .....                                       | 145        |
| 12.6. Diverging integrals .....                                                             | 146        |
| <b>13. Integral geometry in Lorentz spaces .....</b>                                        | <b>147</b> |
| 13.1. A Cauchy formula in the Lorentz plane .....                                           | 148        |
| 13.2. Other Lorentz manifolds .....                                                         | 149        |
| <b>14. The space of spheres .....</b>                                                       | <b>153</b> |
| 14.1. Lorentz-Minkowski space and spheres .....                                             | 153        |
| 14.2. Spheres of dimension 0 .....                                                          | 156        |
| 14.3. Points and circles in $\mathbb{S}^2$ .....                                            | 159        |
| 14.4. Spheres of dimension two in $\mathbb{S}^3$ .....                                      | 162        |
| <b>15. Integral geometry in hyperbolic spaces .....</b>                                     | <b>169</b> |
| 15.1. Measure on the set of geodesics of $\mathbb{H}^2$ .....                               | 169        |
| 15.2. Integral geometry of boundaries of compact convex domains and<br>surfaces .....       | 172        |
| 15.3. Integral geometry in compact hyperbolic surfaces .....                                | 175        |
| 15.4. Integral geometry with horocycles .....                                               | 176        |
| <b>16. The spherical analogue of tightness .....</b>                                        | <b>179</b> |



|                                                                                                   |     |
|---------------------------------------------------------------------------------------------------|-----|
| 16.1. The circle two-piece property .....                                                         | 179 |
| 16.2. The spherical two-piece property .....                                                      | 181 |
| <b>17. Conformal integral geometry</b> .....                                                      | 187 |
| 17.1. Some conformal invariants of a knot or a link .....                                         | 187 |
| 17.2. Spheres and two-component links .....                                                       | 191 |
| 17.3. A conformal (Fary-Fenchel-Milnor)-like theorem .....                                        | 199 |
| 17.4. Intersection of surfaces of the sphere $\mathbb{S}^3$ with spheres .....                    | 206 |
| <b>18. Conformal integral geometry of foliations</b> .....                                        | 209 |
| 18.1. A local conformal invariant of 3-dimensional foliation of $\mathbb{R}^3$ .....              | 209 |
| 18.2. Local and bilocal invariants of plane foliations .....                                      | 212 |
| <b>19. Complex integral geometry</b> .....                                                        | 219 |
| 19.1. Set of lines .....                                                                          | 220 |
| 19.2. Cauchy-Crofton formula in $\mathbb{C}^2$ .....                                              | 221 |
| 19.3. Local geometry of complex curves in $\mathbb{C}^2$ .....                                    | 222 |
| 19.4. Critical points of projections on complex lines and the complex Gauss<br>map .....          | 224 |
| <b>20. Appendix</b> .....                                                                         | 235 |
| 20.1. Archimedes computation of an area .....                                                     | 235 |
| 20.2. Measure associated with a Riemannian metric .....                                           | 237 |
| 20.3. Curvature associated to a Riemannian metric on a surface .....                              | 237 |
| 20.4. The Brouwer degree of a map and index of an isolated singularity .....                      | 238 |
| 20.5. Critical points of a function .....                                                         | 239 |
| 20.6. Generic properties .....                                                                    | 240 |
| 20.7. Coarea formula .....                                                                        | 240 |
| 20.8. Foliations .....                                                                            | 242 |
| 20.9. Fiber bundles .....                                                                         | 245 |
| 20.10. Hopf fibration .....                                                                       | 246 |
| 20.11. Linear pencils .....                                                                       | 247 |
| 20.12. Models of the hyperbolic plane .....                                                       | 250 |
| 20.13. Dupin cyclides .....                                                                       | 252 |
| 20.14. Complements and conjectures in conformal geometry .....                                    | 254 |
| 20.15. Topology .....                                                                             | 259 |
| 20.16. The Poincaré-Hopf theorem; an extension for non-orientable foliations<br>of the disc ..... | 263 |
| <b>Index of notations</b> .....                                                                   | 265 |
| <b>Index of names</b> .....                                                                       | 267 |
| <b>Index</b> .....                                                                                | 269 |
| <b>Bibliography</b> .....                                                                         | 273 |

