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A random walk among random graphs

Nicolas CURIEN

31

A RANDOM WALK
AMONG RANDOM GRAPHS

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Abstract. — Random graphs now stand at the forefront of modern probability and statistics, serving as powerful tools for modeling complex systems across disciplines. This course, tailored for Master's and PhD students, offers a rigorous and insightful introduction to the foundational models of random graph theory—among them the Bienaymé-Galton-Watson trees, the Erdős-Rényi graph, and preferential attachment models such as the Barabási-Albert graph. We present short and modern proofs of landmark results, including the emergence of a giant component in Erdős-Rényi graphs and the asymptotic behavior of distances and degree distributions in preferential attachment networks. Special emphasis is placed on the core probabilistic techniques that drive these analyses—such as the method of moments, random walk theory, and Poissonization—equipping students with versatile tools that apply far beyond the scope of this course.

Résumé (*Une promenade parmi des graphes aléatoires*). — Les graphes aléatoires occupent une place centrale en probabilité et en statistique. Ce sont de puissants outils pour modéliser des systèmes complexes dans de nombreux domaines. Ce cours, conçu pour des étudiants de master et de doctorat, propose une introduction aux modèles fondamentaux de la théorie des graphes aléatoires – parmi lesquels les arbres de Bienaymé-Galton-Watson, le graphe d'Erdős-Rényi, ainsi que les modèles à attachement préférentiel, tels que le graphe de Barabási-Albert. Nous présentons des démonstrations modernes et concises de résultats majeurs, comme l'émergence d'une composante géante dans le graphe d'Erdős-Rényi, ou encore le comportement asymptotique des degrés et des distances dans les graphes à attachement préférentiel. Une attention particulière est portée aux outils probabilistes fondamentaux qui sous-tendent ces résultats – notamment la méthode des moments, la théorie des marches aléatoires et la poissonnisation – dotant ainsi les étudiants d'un ensemble de techniques puissantes, applicables bien au-delà du champ de ce cours.

CONTENTS

A RANDOM WALK AMONG RANDOM GRAPHS

The theory of random graphs is now ubiquitous in probability theory, and there are already many comprehensive textbooks (to name just a few [**113, 114, 23, 69, 46, 49**]) dealing with the numerous models of random graphs invented over the last decades. The goal of these lecture notes is to give a glimpse of a few models of random graphs together with some of the probabilistic tools used to study them. It is intended for master or PhD students in probability theory. I chose the models of random graphs mainly by taste and by the will to cover different types of probabilistic arguments. This document should not be seen as an authoritative reference but rather as a recreational (random) walk in the wonderland of random graph theory. Several exercises of varying difficulty (most of them being non trivial) are scattered along the text and each chapter is ended with bibliographical pointers. Here are the main topics covered in these lecture notes together with the *mathematical tools they introduce*:

- Chapter I: Basic of (bond) percolation. Phase transition. The Rado graph.
  Graph theory, first and second moment, duality.
- Chapter II: One-dimensional random walk, Recurrence/transience, Oscillation/drift.
  Law of large numbers and its reciproque, Fourier transform.
- Chapter III: Skip-free random walk, duality and cycle lemma. Applications: Kemperman formula, Ballot theorem, parking on the line.
  Feller combinatorial cyclic lemma.
- Chapter IV: Bienaymé-Galton-Watson trees, Łukasiewicz encoding, Enumeration.
  Formal series, Neveu's plane tree formalism.
- Chapter V: Sharp threshold for graph properties on the Erdős-Rényi: connectedness, clique number, diameter, cycle. Convergence of the spectrum.
  First and second moment method, method of moments, Poisson paradigm.
- Chapter VI: Phase transition for the giant component I.
  ε-cut, first moment method, sprinkling, multiplicative coalescent.
- Chapter VII: Phase transition for the giant component II.
  Markovian exploration, differential equation method.
- Chapter VIII: Phase transition for the giant component III.
  Poissonization, bin counting processes, Brownian asymptotics.

- CHAPTER IX: (Uniform) random permutations. Poisson-Dirichlet distribution and Dyckman function for large cycles. Poisson limit for small cycle counts.
  Feller's coupling, randomization, stick breaking construction.
- CHAPTER X: Random recursive tree (and random permutations).
  Chinese restaurant process, recursive distributional equation, Pólya urn scheme.
- CHAPTER XI: Continuous-time embedding and applications.
  Athreya-Karling embedding of Markov chains, convergence of Yule processes and links between exponential and Poisson processes.
- CHAPTER XII: Spine decomposition and applications.
  Martingale transform, spine decomposition, many-to-one formulas.
- CHAPTER XIII: Barabási-Albert random tree.
  Preferential attachment mechanism, scale-free random networks.

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N.C.