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FERMAT'S METHOD OF QUADRATURE

Jaume Paradís, Josep Pla & Pelegrí Viader

Abstract. — The *Treatise on Quadrature* of Fermat (c. 1659), besides containing the first known proof of the computation of the area under a higher parabola, $\int x^{+m/n}dx$, or under a higher hyperbola, $\int x^{-m/n}dx$—with the appropriate limits of integration in each case—has a second part which was mostly unnoticed by Fermat's contemporaries. This second part of the *Treatise* is obscure and difficult to read. In it Fermat reduced the quadrature of a great number of algebraic curves in implicit form to the quadrature of known curves: the higher parabolas and hyperbolas of the first part of the paper. Others, he reduced to the quadrature of the circle. We shall see how the clever use of two procedures, quite novel at the time: the change of variables and a particular case of the formula of integration by parts, provide Fermat with the necessary tools to square—quite easily—as well-known curves as the folium of Descartes, the cissoid of Diocles or the witch of Agnesi.

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RÉSUMÉ (La méthode de Fermat pour les quadratures). — Le *Traité des quadratures* de Fermat (vers 1659), contient, outre la première démonstration connue du calcul de l'aire sous une parabole supérieure, $\int x^{+m/n}dx$, ou sous une hyperbole supérieure, $\int x^{-m/n}dx$—avec les limites d'intégration correspondants a chaque cas—, une seconde partie qui est passée presque inaperçue aux yeux de ses contemporains. Cette partie du *Traité* est obscure et difficile à lire. Fermat y réduit la quadrature d'un grand nombre de courbes algébriques données sous forme implicite à la quadrature connue de certaines courbes: les paraboles et hyperboles de la première partie de son article. D'autres quadratures sont obtenues par réduction à la quadrature du cercle. Nous verrons comment l'usage intelligent de deux procédés, assez nouveaux à l'èpoque, le changement de variables et un cas particulier de la formule d'intégration par parties, en fait un outil pour quarrer—assez facilement—des courbes aussi fameuses que le folium de Descartes, la cissoïde de Dioclès et la cubique (sorcière) d'Agnesi.

1. INTRODUCTION

One of the last papers of Fermat is devoted to the quadrature (in the sense of finding the area of a plane region enclosed by a curve and some other lines) of a wide family of algebraic curves, among which the best known and more widely treated by historians are the "higher parabolas", that is curves with equations of the form $y = x^{m/n}$, with m, n integers and $m/n > 0$ and "higher hyperbolas", with the same equation but with $m/n < 0, \neq -1$. This is done in the first part of Fermat's paper. The second part is concerned with the reduction of the quadrature of some curves to the quadrature of others. The paper was written around 1659 and has quite a lengthy title:

> On the transformation and alteration of local equations for the purpose of variously comparing curvilinear figures among themselves or to rectilinear figures, to which is attached the use of geometric proportions in squaring an infinite number of parabolas and hyperbolas. [Translation by Mahoney, Mahoney 1994, p. 245]. [1]

This long title, understandably enough, has been abridged to *Treatise on Quadratures* [*ibid*].

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[1] *De æquationum localium transmutatione et emendatione ad multimodam curvilineorum inter se vel cum rectilineis comparationem, cui annectitur proportionis geometricæ in quadrandis infinitis parabolis et hyperbolis usus* [Fermat c. 1659, p. 255].

It was published in 1679 as part of the collected works of Fermat edited by his son, Clément-Samuel [Fermat 1679, pp. 44–57].

Prior to the *Treatise*, Fermat had done some work on quadratures. He had tried, unsuccessfully, to square the cycloid [2] and some correspondence with Cavalieri and Torricelli proves that he had already been working on the problem of quadratures at the end of the 1630's and early 1640's, see [Mahoney 1994, p. 244].

A description of the contents of the *Treatise* can be found in two important works: Zeuthen's [1895] and Mahoney's [1994]. Mahoney's is a book published originally in 1973 with a second printing in 1994 and can be considered as the current obliged reference on Fermat's mathematical work.

In 1644, according to [Zeuthen 1895, pp. 41–45], Fermat was already in possession of the proof of the computation of the quadrature on $[0, b]$ of the parabolas with equation $b^m y^n = b^n x^m$, with m, n positive integers, and b a given constant. [3] This was precisely the year in which Fermat sent his results to Cavalieri via father Mersenne. The complete transcription of his work on quadrature into the *Treatise* must have taken place after 1657, most likely in 1659 [Mahoney 1994, pp. 244–245, 421], [Zeuthen 1895, p. 45]. In that same year he included the quadrature on $[b, \infty)$ of the higher hyperbolas $x^m y^n = b^{n+m}, \quad m > n,$ [4] using an appropriate partition of the coordinate axes with the help of geometrical progressions. The details can be found in [Mahoney 1994, pp. 245–254], [Bos et al. 1980; Boyer 1945; Katz 1993]. Respect this part of the *Treatise* we have nothing new to say. It has been thoroughly studied for its great importance within the history of integration since it goes apace with the research of other mathematicians of the 17$^{\text{th}}$ century as Pascal, Cavalieri, Torricelli, Wallis, Barrow, etc. who were working on the problem of the integration of x^n .

2 This is a problem that Wallis solves in his *Tractatus duo. Prior de cycloide [...]* (1659) using his method of "interpolation by analogy", see [Whiteside 1961, pp. 242-243]. The first to square the cycloid was Roberval in 1634 in his *Traité des indivisibles,* (first published in Paris 1693), see [Walker 1932].

3 Fermat multiplies each side of the equation by the constant b raised to the necessary power in order to maintain the homogeneity of dimensions. See later note 15.

4 With the exception $xy = b^2$.