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CANONICAL HOMOTOPY OPERATORS FOR THE $\bar{\partial}$ COMPLEX IN STRICTLY PSEUDOCONVEX DOMAINS

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ABSTRACT. — In a strictly pseudoconvex domain $D = \{\rho < 0\}$ in $\mathbb{C}^n$, we study the homotopy operators K_α for $\bar{\partial}$ that are canonical with respect to the metric $(-\rho)i\partial\bar{\partial}\log(1/-\rho)$ and weights $(-\rho)^\alpha$, $\alpha > 0$, and their relation to the canonical homotopy operator K_b for $\bar{\partial}_b$ on ∂D . We prove that the boundary values of the kernel of K_α in the ball are provided by well-known integral formulas due to Henkin, Skoda et al. We are able to compute the kernel for K_α in the interior of D , by using a technique for representing forms in D by complex tangential forms on the boundary of a higher dimensional domain. This is a generalization of a well-known technique for functions. In the ball we also prove the commutation rule $\partial/\partial z_\ell K_\alpha = K_{\alpha+1}\partial/\partial \zeta_\ell$, which generalizes a well-known fact about the weighted Bergman projections, and use it to construct homotopy formulas for $\partial\bar{\partial}$ in the ball.

RÉSUMÉ. — OPÉRATEURS D'HOMOTOPIE DANS LES DOMAINES STRICTEMENT PSEUDO-CONVEXES. — Dans un domaine strictement pseudoconvexe $D = \{\rho < 0\}$ dans $\mathbb{C}^n$, on étudie les opérateurs d'homotopie K_α pour les $\bar{\partial}$ canoniques pour la métrique $(-\rho)i\partial\bar{\partial}\log(1/-\rho)$ et les poids $(-\rho)^\alpha$, $\alpha > 0$ et leur relation avec l'opérateur canonique d'homotopie K_b pour le $\bar{\partial}_b$ dans ∂D . On démontre que les valeurs au bord du noyau de K_α pour la boule sont donnés par les formules intégrales de Henkin, Skoda et al. On parvient à calculer le noyau de K_α à l'intérieur du domaine D en utilisant une technique pour représenter les formes dans D par des formes tangentielle complexes au bord d'un domaine dans une dimension plus élevée. Il s'agit là d'une généralisation

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d'une technique bien connue dans les cas des fonctions. Dans la boule, on prouve aussi la loi de commutation $\partial/\partial z_\ell K_\alpha = K_{\alpha+1} \partial/\partial \zeta_\ell$, qui généralise un résultat déjà connu des projections de Bergman à poids. On utilise ce fait pour construire des formules d'homotopie pour le $\partial\bar{\partial}$ dans la boule.

1. Introduction

Estimates of growth and regularity of solutions to the $\bar{\partial}$ -equation in domains in $\mathbb{C}^n$ were obtained by L^2 -methods in the 1960's by Kohn, Hörmander et al. In the 1970's Henkin, Skoda, and others introduced formulas for representation of solutions that gave further information such as L^p -estimates, Hölder estimates, and so on. These two methods have been living side by side but the interplay between them is not fully understood. In this paper we offer a geometrical interpretation of the well-known Henkin-Skoda solution formula for $\bar{\partial}$ and its weighted analogues in strictly pseudoconvex domains. This leads to mutual exchange of information between the explicit formulas and L^2 -theory. For instance, regularity properties of the abstractly defined operators can be derived from the explicit formulas, and in the other direction, by studying the adjoint of the $\bar{\partial}$ -operator, we can prove a certain commutation property of the kernels and derivatives in the case of the ball, which makes it possible to construct explicit homotopy formulas for $\partial\bar{\partial}$.

The Henkin-Skoda solution formula provides the L^2 -minimal solution for $\bar{\partial}$ in the ball but it does not coincide as an operator with the Kohn operator, which is the one that gives the minimal solution when applied to a $\bar{\partial}$ closed form and vanishes on forms that are orthogonal to the kernel of $\bar{\partial}$, with respect to the Euclidean L^2 norm. In [16], Harvey and Polking found an explicit expression for the Kohn operator, essentially expressed by rational functions, but anyway not as simple as the Henkin-Skoda formula. In order to interpret the Henkin-Skoda formula geometrically we have to introduce norms closely related to the Bergman norm.

Throughout this paper we let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, and ρ a strictly plurisubharmonic defining function. For $\alpha > 0$, let L_α^2 be the space of locally square integrable forms in D such that

$$(1.1) \quad \|f\|_\alpha^2 = \frac{\Gamma(n+\alpha)}{2^n \pi^n \Gamma(\alpha)} \int_D (-\rho)^\alpha |f|^2 dV$$

is finite, where $|f|$ is the norm of the form f with respect to the metric defined by the form

$$(1.2) \quad \Omega = (-\rho) i \partial \bar{\partial} \log \left(\frac{1}{-\rho} \right)$$

(in the case of the ball Ω is just distance to the boundary times the Bergman metric) and $dV = \Omega^n/n!$. The volume form dV is equivalent to $(1/\rho)$ times the Lebesgue measure, and therefore $A_\alpha^2 = L_\alpha^2 \cap \mathcal{O}(D)$ is the usual Bergman space with weight $(-\rho)^{\alpha-1}$. We let K_α be the operator on L_α^2 defined so that $K_\alpha f$ is the minimal solution to $\bar{\partial}u = f$ if f is a $\bar{\partial}$ -closed $(0, q+1)$ -form in D (the existence of such a solution is well-known, see Theorem 2.2) and $K_\alpha f = 0$ if f is orthogonal to $\mathcal{K}_\alpha = L_\alpha^2 \cap \text{Ker } \bar{\partial}$. If P_α is the orthogonal projection of functions in L_α^2 onto A_α^2 , the Bergman projection, then the relation

$$(1.3) \quad K_\alpha \bar{\partial} f + \bar{\partial} K_\alpha f = f - P_\alpha f,$$

holds for f in $\text{Dom } \bar{\partial}$. (It is enough to verify it separately for $f \in \mathcal{K}_\alpha$ and $f \in \mathcal{K}_\alpha^\perp \cap \text{Dom } \bar{\partial}$, and both these cases follow immediately from the definition.)

These operators are natural to study for several reasons. To begin with, it is well-known that the $\bar{\partial}$ -operator behaves like half a derivative in the complex tangential directions near the boundary of a strictly pseudoconvex domain. This is reflected in the standard estimates for $\bar{\partial}$. For instance, the well-known Henkin-Skoda estimate, [19] and [21], states that $\bar{\partial}u = f$ has a solution (f being a $\bar{\partial}$ -closed $(0, q+1)$ -form) such that

$$(1.4) \quad \int_{\partial D} |\bar{\partial}\rho \wedge u|_E \leq C \int_D (-\rho)^{-1/2} [\sqrt{-\rho} |f|_E + |\bar{\partial}\rho \wedge f|_E].$$

Here, $|\cdot|_E$ denotes the Euclidean norm of a form, and since $-\rho$ is approximately the distance to the boundary, $\bar{\partial}\rho \wedge f$ determines the complex tangential part of f near the boundary. It is well-known that this boundary behaviour of $\bar{\partial}$ is reflected by the Bergman metric and therefore by Ω as well.

The estimate (1.4) was the first important success for weighted integral formulas; once they are constructed the estimate follows nicely, as the very feature of the formulas reflects this difference in normal and complex tangential part. This suggests that these operators better should be understood in terms of a metric like Ω that takes this difference into account.

One of our main results (Theorem 5.1) is that in the ball case, the boundary values of the operator due to Henkin and Skoda, and its weighted analogues, in fact coincide with (the boundary values of) the canonical operator K_α . Expressed in the inner product $(\cdot, \cdot)_\alpha$ connected to $\|\cdot\|_\alpha$, the kernel for the boundary values of K_α has the simple expression

$$(1.5) \quad k_\alpha(\zeta, z)$$

$$= \sum_{q=0}^{n-1} \frac{\Gamma(\alpha + n - q - 1)}{\Gamma(n + \alpha)} \frac{\bar{z} \cdot d\zeta \wedge (d\bar{z} \cdot d\zeta)^q}{(1 - \bar{\zeta} \cdot z)^{\alpha-1+n-q} (1 - \zeta \cdot \bar{z})^{q+1}}, \quad z \in \partial\mathbb{B}.$$

In the general strictly pseudoconvex case one cannot hope for completely explicit formulas for K_α . However, it turns out that well-known solution operators approximately provide the boundary values of K_α , in much the same spirit that certain known explicit holomorphic projection operators approximate the Bergman and the Szegő projections in D , as was proved by Kerzman-Stein, [17], see [4] and [5].

The spaces L_α^2 and operators K_α have connections to the boundary complex. In Section 2 we notice that $\|f\|_\alpha$ tends to $\|f\|_b$, the L^2 -norm of the complex tangential part $f|_b$ of f over ∂D , when $\alpha \rightarrow 0$. It is also true that (the boundary values of) K_α tend to K_b , the canonical operator for $\bar{\partial}_b$. In the ball case this follows from Section 5, whereas the general case is treated in [4], [5].

There is another useful connection to the boundary complex that is exploited in Section 4. It is well-known and used by many authors the fact that if

$$\tilde{D} = \{(z, w) \in \mathbb{C}^{n+1}; \rho(z) + |w|^2 < 0\},$$

then $L^2(D)$ can be identified with the subspace $L^2(\partial\tilde{D})$ consisting of functions that are rotation invariant in the last variable, and that furthermore, *via* this identification, the orthogonal projection onto the Bergman space A^2 in D , the Bergman projection, corresponds to the Szegő projection on $\partial\tilde{D}$. We extend this representation to higher order forms, so that forms in D correspond to certain tangential forms on $\partial\tilde{D}$ and so that the orthogonal projection of forms in L_1^2 onto $\mathcal{K}_1$ corresponds to the orthogonal projection of L_b^2 onto $\mathcal{K}_b = L_b^2 \cap \text{Ker } \bar{\partial}_b$ on $\partial\tilde{D}$. We also show that the canonical operator K_α can be represented by the complex tangential boundary values of the corresponding $\tilde{K}_{\alpha-1}$ in $\tilde{D}$. Since well-known formulas in the ball give the boundary values, we therefore get an effective procedure to compute the values of K_α in the interior. This is the done in Section 5. The resulting formulas have not previously occurred in the literature (as far as we know).

A basic ingredient in our proofs is the formula for the formal adjoint $\bar{\partial}_\alpha^*$, that is computed in Section 3. It is a first order differential operator with coefficients that are smooth up to the boundary. It turns out that any smooth f belongs to the domain $\text{Dom } \bar{\partial}_\alpha^*$ of the von Neumann adjoint of $\bar{\partial}$, and for $\alpha \geq 1$, $f \in \text{Dom } \bar{\partial}_\alpha^*$ if and only if $f, \bar{\partial}_\alpha^* f \in L_\alpha^2$.

It is known since long ago that in the ball

$$\frac{\partial}{\partial z_j} P_\alpha f = P_{\alpha+1} \left(\frac{\partial f}{\partial \zeta_j} \right).$$