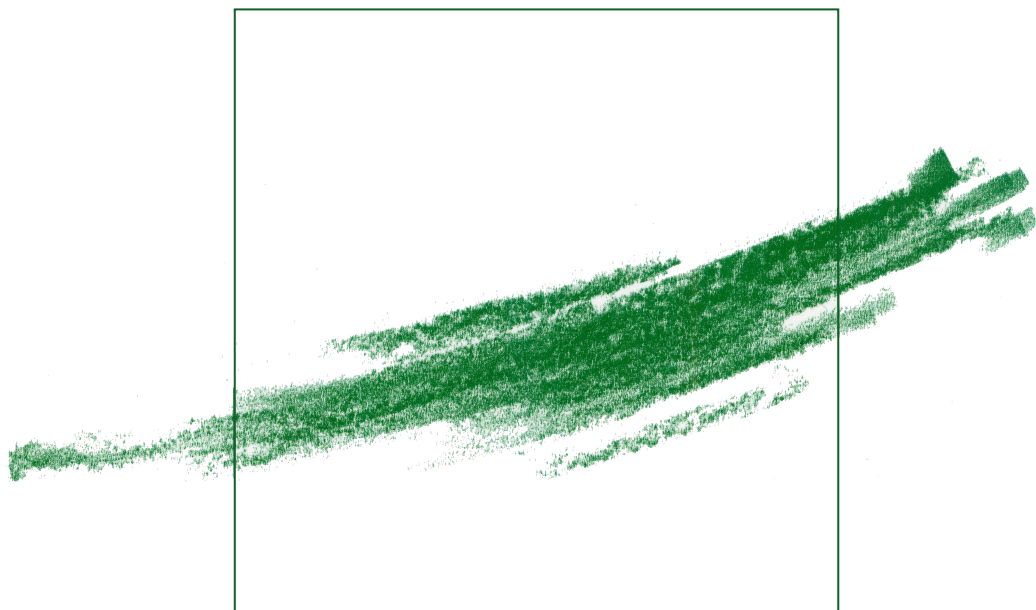


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# Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23



# **INTEGRAL GEOMETRY FROM BUFFON TO GEOMETERS OF TODAY**

**Rémi Langevin**

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**Rémi Langevin**

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