

CENTRAL SHEAVES ON AFFINE FLAG VARIETIES

Pramod N. Achar & Simon Riche



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CENTRAL SHEAVES ON AFFINE FLAG VARIETIES

Pramod N. Achar & Simon Riche

Abstract. — In this book we explain the construction (due to Gaitsgory) of the “central sheaves” on the affine flag variety of a reductive algebraic group associated with the representations of the Langlands dual group. These objects are the categorical counterpart of the Bernstein description of the center of the affine Hecke algebra. They play a crucial role in many constructions related to the geometric Langlands program, some of which have important applications to the representation theory of reductive algebraic groups and associated quantum groups. We also explain the proof of the main properties of these objects (in particular the filtration by Wakimoto sheaves), for arbitrary coefficients. Finally, we present in detail the construction of an equivalence of categories due to Arkhipov-Bezrukavnikov relating certain derived categories of constructible sheaves on the affine flag variety and coherent sheaves on the Springer resolution of the dual group, in which central sheaves play a key role.

Résumé. (Faisceaux centraux sur les variétés de drapeaux affines) — Dans ce livre nous expliquons la construction (due à Gaitsgory) des « faisceaux centraux » sur la variété de drapeaux affine d’un groupe algébrique réductif associés aux représentations du groupe dual au sens de Langlands. Ces objets sont le pendant « catégorique » de la description de Bernstein du centre de l’algèbre de Hecke affine. Ils jouent un rôle crucial dans de nombreuses constructions liées au programme de Langlands géométrique, dont certaines ont des applications importantes en théorie des représentations des groupes algébriques réductifs et des groupes quantiques associés. Nous expliquons également la preuve des propriétés principales de ces objets (notamment la filtration par les objets de Wakimoto), pour des coefficients arbitraires. Enfin, nous présentons en détail la construction d’une équivalence de catégories due à Arkhipov-Bezrukavnikov qui relie certaines catégories dérivées de faisceaux constructibles sur la variété de drapeaux affine et de faisceaux cohérents sur la résolution de Springer du groupe dual, et dans laquelle les faisceaux centraux jouent un rôle crucial.

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