

GROUPS ACTING ON FRACTALS, HYPERBOLICITY AND SELF-SIMILARITY

François Dahmani, Anna Erschler,
Camille Horbez, Daniel Wise (eds.)



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Abstract. — This volume collects the lecture notes of the mini-courses given during the special trimester *Groups Acting on Fractals, Hyperbolicity and Self-Similarity*, held at the Institut Henri Poincaré in April-July 2022. Its guiding principle was fractalness in group theory, articulated around three main topics: self-similar groups acting on rooted trees, hyperbolic groups and their boundaries, and automorphism groups of free groups and surface groups. We hope this collection of mini-courses will highlight the rich and fruitful interactions between these topics, and provide a resource for a renewed community of researchers in the field.

Résumé. (Groupes agissant sur des fractals, hyperbolicité et auto-similarité) — Ce volume rassemble les notes des mini-cours donnés lors du trimestre spécial *Groupes agissant sur des fractales, hyperbolicité et auto-similarité*, qui s'est tenu à l'Institut Henri Poincaré d'avril à juillet 2022. Son principe directeur était la fractalité dans les groupes, articulée autour de trois principaux thèmes : les groupes auto-similaires agissant sur des arbres enracinés, les groupes hyperboliques et leurs bords, et les automorphismes de groupes libres et de groupes de surfaces. Nous espérons que ce recueil de mini-cours soulignera les interactions riches et fructueuses entre ces thèmes, et constituera une ressource pour une communauté sans cesse renouvelée de personnes travaillant dans ce domaine.

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Mini-course on the embedding problem between RAAGs

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Given a graph Γ , the *right-angled Artin group* $A(\Gamma)$, also referred to as a *partially commutative group*, is given by the presentation

$$\langle u \in V(\Gamma) \mid [u, v] = 1, \{u, v\} \in E(\Gamma) \rangle.$$

This mini-course is dedicated to the Embedding Problem in right-angled Artin groups, namely: given two finite graphs Φ, Ψ , how to determine whether or not $A(\Phi)$ is isomorphic to a subgroup of $A(\Psi)$? We describe a geometric framework, based on the geometry of *quasi-median graphs*, in which this question can be studied.