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INTERTWINING OPERATORS IN THE TAKEDA–WOOD ISOMORPHISM

by Fei Chen & Wen-Wei Li

Abstract. — Over any non-Archimedean local field of characteristic not equal to 2, Takeda and Wood constructed types for the two blocks containing the even and odd Weil representations of the metaplectic group $\tilde{G}$ and identified the resulting Hecke algebras $H^{\pm}_{\psi}$ with the Iwahori–Hecke algebras of odd orthogonal groups $G^{\pm}$ of the same rank. We describe normalized parabolic induction and Jacquet modules in terms of Hecke modules using a suitable variant of Bushnell–Kutzko theory. Furthermore, we match the standard intertwining operators of $\tilde{G}$ and $G^{\pm}$ by proving a variant of Gindikin–Karpelevich formula for $\tilde{G}$. As an application, we describe the behavior of normalized intertwining operators of $\tilde{G}$ in these blocks under Aubert involution, reducing everything to the $G^{\pm}$ side. This is mainly motivated by Arthur's local intertwining relations.

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Résumé (*Opérateurs d'entrelacement dans l'isomorphisme de Takeda-Wood*). — Pour tout corps local non-archimédien de caractéristique inégale à 2, Takeda et Wood ont construit les types pour les deux blocs contenant les représentations de Weil paire et impaire du groupe métaplectique $\tilde{G}$, et identifié les algèbres de Hecke $H_{\psi}^{\pm}$ correspondantes aux algèbres d'Iwahori–Hecke des groupes orthogonaux $G^{\pm}$ du même rang. Nous décrivons les induites paraboliques normalisées et les modules de Jacquet en termes de modules de Hecke en utilisant une variante convenable de la théorie de Bushnell–Kutzko. De plus, nous relions les opérateurs d'entrelacement standards de $\tilde{G}$ et $G^{\pm}$ en donnant une variante de la formule de Gindikin–Karpelevich pour $\tilde{G}$. Par conséquent, nous décrivons le comportement sous l'involution d'Aubert des opérateurs d'entrelacement normalisés dans ces blocs, en se ramenant au côté de $G^{\pm}$. Ceci est motivé par les relations d'entrelacement locales d'Arthur.

1. Introduction

This work has grown out of a desire to understand Arthur's *local intertwining relations* (LIR) for metaplectic coverings, whose formulation for quasisplit classical groups can be found in [2, §2.4]. LIR is one of the pillars of Arthur's conjectures in the Langlands program. Roughly speaking, it is a statement about the relation between intertwining operators and endoscopy, or the structure of Arthur packets under parabolic induction. The tempered LIR for metaplectic groups is settled in [11] by reducing to odd orthogonal groups via θ-lift. As for the non-tempered case, if one tries to adopt Arthur's strategy in [2, Chapter 7], some initial knowledge about LIR would be indispensable for bootstrapping the local-global argument. Such knowledge includes, but is not limited to the "unramified case", and one needs results alike at *every* non-Archimedean place. This makes the metaplectic LIR for metaplectic groups more delicate than the classical counterparts, since one has to deal with $p = 2$.

The aim of this work is to understand intertwining operators for genuine representations in the two simplest Bernstein blocks of metaplectic group over a non-Archimedean local field F of *arbitrary* residual characteristic p, namely the blocks containing the even and odd Weil representations, respectively. They are the metaplectic analogues of Iwahori-spherical Bernstein blocks. The hope is to transfer the problems to odd orthogonal groups as much as possible.

To fix notations, let $\mathfrak{o}$ (resp. ϖ) be the ring of integers (resp. a uniformizer) of F, and let $|\cdot|_F$ be the normalized absolute value on F. Let $(W, \langle\cdot|\cdot\rangle)$ be a symplectic F-vector space with prescribed symplectic basis $e_1, \ldots, e_n, f_n, \ldots, f_1$, whence the standard Borel pair $(B^{\rightarrow}, T)$. The corresponding symplectic group $\mathrm{Sp}(W)$ is identified with its F-points. Fix an additive character ψ of F such that

$$\psi_{4\mathfrak{o}} = 1, \quad \psi_{4\varpi^{-1}\mathfrak{o}} \neq 1.$$

Let $\boldsymbol{\mu}_m$ be the group of m-th roots of 1 in $\mathbb{C}$. The twofold metaplectic covering is a central extension

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{Mp}(W) \xrightarrow{\mathbf{p}} \mathrm{Sp}(W) \to 1$$

of locally compact groups. It arises from the Heisenberg group $H(W) = W \times F$ with multiplication $(w,t)(w',t') = (w+w', t+t'+\langle w|w'\rangle)$.

Following [16] and its sequels, we will push the covering out along $\boldsymbol{\mu}_2 \hookrightarrow \boldsymbol{\mu}_8$ to obtain the covering

$$1 \to \boldsymbol{\mu}_8 \to \widetilde{\mathrm{Sp}}(W) \xrightarrow{\mathbf{p}} \mathrm{Sp}(W) \to 1.$$

It carries the Weil representation $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$ (even $\oplus$ odd), which is genuine. Recall that a representation of $\widetilde{\mathrm{Sp}}(W)$ is genuine if $\boldsymbol{\mu}_8$ acts tautologically.

Let $G := \mathrm{Sp}(W)$ and $\tilde{G} := \widetilde{\mathrm{Sp}}(W)$. The main benefit of eightfold coverings is that given a Levi subgroup $M = \mathrm{Sp}(W^\flat) \times \prod_{i \in I} \mathrm{GL}(n_i)$ of G, where $W^\flat \subset W$ is a symplectic subspace, we have

$$\tilde{M} := \mathbf{p}^{-1}(M(F)) \simeq \widetilde{\mathrm{Sp}}(W^\flat) \times \prod_{i \in I} \mathrm{GL}(n_i, F);$$

the canonical isomorphism above is given by the Schrödinger model for ω_ψ.

Moreover, it is a general property of coverings that the preimage of any parabolic $P = MU \subset G$ splits in $\tilde{G}$ into $\tilde{P} := \mathbf{p}^{-1}(P(F)) = \tilde{M}U(F)$.

We are interested in the genuine smooth representations of $\tilde{G}$. The category of such representation decomposes into Bernstein blocks. Let $\mathcal{G}_\psi^\pm$ be the blocks containing $\omega_\psi^\pm$.

Following the work [9] of Gan and Savin (for $p > 2$), Takeda and Wood [23] constructed types for $\mathcal{G}_\psi^\pm$ in the sense of Bushnell–Kutzko [6]. They gave presentations for the corresponding Hecke algebras $H_\psi^\pm$ and established an isomorphism

$$\mathrm{TW} : H^\pm \xrightarrow{\sim} H_\psi^\pm$$

as Hilbert algebras. Here, $H^\pm = H^{G^\pm}$ is the Iwahori–Hecke algebra for $G^\pm := \mathrm{SO}(V^\pm)$, where $V^\pm$ is a quadratic F-vector space of dimension $2n+1$, discriminant 1 and Hasse invariant ± 1. Consequently, $\mathcal{G}_\psi^\pm \simeq \mathcal{G}^\pm$, where $\mathcal{G}^\pm$ denotes the Iwahori-spherical block for $G^\pm$.

Since we have split Levi subgroups $\tilde{M} \subset \tilde{G}$ into a smaller metaplectic group and GL-factors, the results above extend to $\tilde{M}$ and the corresponding Levi subgroups $M^\pm \subset G^\pm$ (i.e., with the same pattern of $\mathrm{GL}(n_i)$'s): we have $\mathrm{TW} : H^{M^\pm} \xrightarrow{\sim} H_\psi^{\tilde{M},\pm}$.

For representations in $\mathcal{G}^{\pm}_{\psi}$, we wish to translate questions about intertwining operators to the $G^{\pm}$-side through TW. The following issue has to be resolved first.

> How can we describe normalized Jacquet modules $r_{\tilde{P}}$ and parabolic inductions $i_{\tilde{P}}$ in terms of Hecke modules?

For the groups $G^{\pm}$ and the blocks $\mathcal{G}^{\pm}$, the corresponding question is solved in [6] (see also [5]): there is a canonical embedding $\mathrm{t}^{\pm}_{\mathrm{nor}} : H^{M^{\pm}} \to H^{\pm}$ of Hilbert algebras, through which $r_{P^{\pm}}$ matches $(\mathrm{t}^{\pm}_{\mathrm{nor}})^* : H^{\pm}\text{-}\mathsf{Mod} \to H^{M^{\pm}}\text{-}\mathsf{Mod}$, where $P^{\pm} = M^{\pm}U^{\pm} \subset G^{\pm}$ is parabolic. By adjunction, $i_{P^{\pm}}$ matches $\mathrm{Hom}_{H^{M^{\pm}}}(H^{\pm}, \cdot)$.

On the metaplectic side, the problem is that the type constructed by Takeda–Wood for $\tilde{G}$ is NOT a *cover* of that of $\tilde{M}$, in the sense of Bushnell–Kutzko [6, §8], except in the + case with $p > 2$—this can be seen from the fact that the inducing representation in the type for $\tilde{G}$ usually has dimension larger than its counterpart for $\tilde{M}$, for example. Instead, we define the embedding as

$$\mathrm{t}_{\mathrm{nor}} := \text{the composition of } H^{\tilde{M},\pm}_{\psi} \xrightarrow[\sim]{\mathrm{TW}^{-1}} H^{M^{\pm}} \xrightarrow{\mathrm{t}^{\pm}_{\mathrm{nor}}} H^{\pm} \xrightarrow[\sim]{\mathrm{TW}} H^{\pm}_{\psi}.$$

The main results in this article are summarized below. Each of them divides into + (even) and − versions that apply to representations in $\mathcal{G}^{+}_{\psi}$ and $\mathcal{G}^{-}_{\psi}$, respectively. The − case is always more delicate to state and prove.

THEOREM 1.1 (= Theorem 6.3.2 and (6.2), (6.3)). — *Let $P = MU$ be a reverse-standard parabolic subgroup of G. Via the equivalence $\mathbf{M} : \mathcal{G}^{\pm}_{\psi} \xrightarrow{\sim} H^{\pm}_{\psi}\text{-}\mathsf{Mod}$ furnished by type theory and its avatar for $\tilde{M}$, the adjunction pair*

$$r_{\tilde{P}} : \mathcal{G}^{\pm}_{\psi} \rightleftarrows \mathcal{G}^{\tilde{M},\pm}_{\psi} : i_{\tilde{P}}$$

corresponds to

$$\mathrm{t}^{*}_{\mathrm{nor}} : H^{\pm}_{\psi}\text{-}\mathsf{Mod} \rightleftarrows H^{\tilde{M},\pm}_{\psi}\text{-}\mathsf{Mod} : \mathrm{Hom}_{H^{\tilde{M},\pm}_{\psi}}(H^{\pm}_{\psi}, \cdot).$$

The canonical isomorphisms involved in this correspondence are explicit; see Proposition 6.4.2.

Being *reverse-standard* means that $P \supset B^{\leftarrow}$ where $B^{\leftarrow}$ is the Borel subgroup of G containing the standard maximal torus T whose simple roots are

$$\epsilon_{i+1} - \epsilon_i \quad (1 \le i < n), \quad 2\epsilon_1;$$

the same terminology applies to parabolic subgroups of $G^{\pm}$ as well. In contrast, the definition of types for $\mathcal{G}^{\pm}_{\psi}$ uses the standard Iwahori subgroup I of $G(F)$.

The idea of the proof goes as follows. There is a canonical linear map $\mathrm{q} : \mathbf{M}(\pi) \to \mathbf{M}(r_{\tilde{P}}(\pi))$ for any smooth representation π of $\tilde{G}$. Based on the results in [23, §3], one can prove that q is a linear isomorphism, and it is $\mathrm{t_{nor}}$-equivariant up to constants that depend on elements $T_w^{\tilde{M}} \in H_\psi^{\tilde{M},\pm}$ (where w ranges over P-positive elements in the affine Weyl group of M) but not on π. These constants are then pinned down by computing the case of $\pi = \omega_\psi^\pm$. A calculation of co-invariants of types (Proposition 4.2.1) is also needed, and this is why we require P to be reverse-standard.

From this, it is easy to match Aubert involutions between $\mathcal{G}_\psi^\pm$ and $\mathcal{G}^\pm$, for which we refer to [13, §A.2] for a summary.

Theorem 1.2 (= Corollary 10.1.3). — *Let $(\cdot)^\wedge$ denote the Aubert involution as a functor on given Bernstein blocks. The following diagram commutes up to a canonical isomorphism*

$$\begin{tikzcd} \mathcal{G}_\psi^\pm \arrow[r, "\sim"] \arrow[d, "(\cdot)^\wedge"'] & \mathcal{G}^\pm \arrow[d, "(\cdot)^\wedge"] \\ \mathcal{G}_\psi^\pm \arrow[r, "\sim"'] & \mathcal{G}^\pm \end{tikzcd}$$

where the horizontal equivalences are induced by TW.

We proceed to the comparison of standard (non-normalized) intertwining operators. Let Ω_0 (resp. Ω_0^M) be the Weyl groups of G (resp. M).

Theorem 1.3 (= Corollaries 10.2.4 and 10.2.6). — *Let us begin with the + case. Consider reverse-standard parabolic subgroups $P = MU \subset G$ and $P^+ = M^+U^+ \subset G^+$ that match each other in the sense that $M = \mathrm{Sp}(W^\flat) \times \prod_{i=1}^r \mathrm{GL}(n_i)$ and $M^+ = \mathrm{SO}(V^{+,\flat}) \times \prod_{i=1}^r \mathrm{GL}(n_i)$ with $\dim V^{+,\flat} = \dim W^\flat + 1$. Let $w \in \Omega_0$ be such that $wMw^{-1} = M$ and w has minimal length in its Ω_0^M-coset.*

Let π (resp. σ) be of finite length in $\mathcal{G}_\psi^{\tilde{M},+}$ (resp. $\mathcal{G}^{M^+}$) and fix a Hecke-equivariant (relative to TW*) isomorphism between their Hecke modules. Consider the standard intertwining operators*

$$J_w(\pi \otimes \chi) \in \mathrm{Hom}_{\tilde{G}}\left(i_{\tilde{P}}(\pi \otimes \chi), i_{\tilde{P}}({}^{\tilde{w}}(\pi \otimes \chi))\right)$$

viewed as a rational family in unramified characters $\chi : \prod_{i=1}^r \mathrm{GL}(n_i, F) \to \mathbb{C}^\times$, and its avatar $J_w^+(\sigma \otimes \chi)$ for $M^+ \subset G^+$. By interpreting normalized parabolic induction in terms of

$$\mathrm{Hom}_{H_\psi^{\tilde{M},+}}(H_\psi^+, \cdot), \quad \mathrm{Hom}_{H^{M^+}}(H^+, \cdot)$$

using Theorem 1.1, and identifying these spaces via TW*, we have*

$$J_w(\pi \otimes \chi) \xleftrightarrow{\textit{matches}} |2|_F^{t(w)/2} J_w^+(\sigma \otimes \chi)$$

as rational families in χ. Here, $t(w)$ is the number of $B^\leftarrow$-positive long roots $2\epsilon_i$ such that $w(2\epsilon_i)$ is $B^\leftarrow$-negative.