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Pablo Montealegre

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

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ON THE STABLE NORM OF SLIT TORI AND THE FAREY SEQUENCE

by Pablo Montealegre

Abstract. — Let M be a compact manifold endowed with a possibly singular Riemannian metric. The metric induces a norm on the homology of M, called the stable norm. We provide explicit computations of the stable norm of flat slit tori using the Farey sequence. We then glue several slit tori together to produce half-translation surfaces whose unit ball of the stable norm has faces of maximal dimension. Furthermore, we give a subquadratic estimate for the asymptotic counting of simple homology classes on these surfaces.

Résumé (*Sur la norme stable des tores fendus et la suite de Farey*). — Soit M une variété compacte équipée d'une métrique riemannienne que l'on autorise à être singulière. Cette métrique induit une norme sur l'homologie de M, appelée la norme stable. Nous calculons explicitement la norme stable de tores plats fendus à l'aide de la suite de Farey. Ensuite, nous recollons plusieurs tores fendus ensemble pour construire des surfaces de demi-translation dont la boule unité de la norme stable a des faces de dimension maximale. Enfin, nous calculons une estimation asymptotique sous-quadratique du nombre de classes d'homologies simples sur ces surfaces.

Introduction

The stable norm. — Let M be a connected compact manifold endowed with a smooth Riemannian metric. The metric induces a norm on the first homology

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Pablo Montealegre, Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, 34090 Montpellier, France • *E-mail :* `pablo.montealegre@umontpellier.fr`

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group $H_1(M,\mathbb{R})$, named the stable norm by Gromov in [10]. More precisely, following Massart in [15], let $h \in H_1(M,\mathbb{Z})$ be an integral homology class. We define

$$f(h) := \inf_{\gamma} l(\gamma),$$

where the infimum is taken over all the closed rectifiable multicurves γ on M representing the class h, and where $l(\gamma)$ denotes the length of γ. According to [15, Proposition 1.1.3] the limit

$$\lim_{N\to\infty} \frac{f(Nh)}{N}$$

exists for any integral class h; we denote this limit $\|h\|_M$. One can check that for any two integral homology classes h_1 and h_2, the triangle inequality $\|h_1 + h_2\|_M \leq \|h_1\|_M + \|h_2\|_M$ holds. Moreover, for any $N \in \mathbb{Z}$ we have $\|Nh\|_M = |N|\|h\|_M$ so by homogeneity we can naturally extend the function $h \mapsto \|h\|_M$ to the rational homology classes $H_1(M,\mathbb{Q})$. Finally, by a density argument we can extend this function to the whole real homology $H_1(M,\mathbb{R})$, and [15, Proposition 1.1.5] tells us that the extension is a norm.

To this day, little is known about the stable norm of Riemannian manifolds. In particular, there are very few known explicit examples; see Babenko [2] on flat two-dimensional tori, Burago–Ivanov–Kleiner [6] for flat n-dimensional tori, or McShane–Rivin [20] on any punctured hyperbolic torus. The definition of the stable norm remains valid for Riemannian metrics with conical singularities such as translation and half-translation surfaces. These surfaces enjoy very nice geometrical properties; in particular, they are endowed with a flat metric with a finite number of conical singularities (see for instance Zorich's survey [24]) whose cone angles are all integral multiples of π. This greatly simplifies the task of finding geodesics and computing their length, so one can hope to be able to compute explicitly the stable norm on some of these surfaces. Unfortunately, it is still a difficult problem, so the idea of the present paper is to compute the stable norm on flat slit tori that we can later on glue together to obtain translation and half-translation surfaces. This is not new. Since slit tori are the building blocks of surfaces of main interest in modern geometry and dynamical systems they are often a preferential testing ground; see for example [8] or [22].

The slit torus. — A flat slit torus is a compact surface that is homeomorphic to a torus with one boundary component, equipped with a flat Riemannian metric relative to which the boundary of the surface is made of two parallel straight line segments of the same length ρ. Formally, the boundary of the surface can be written as a union $S_1 \cup S_2$, where S_1 and S_2 are homeomorphic to a segment, and $S_1 \cap S_2$ is equal to two distinct points $\{P, P'\}$, such that there exists two local charts φ_1 and φ_2 that map isometrically S_1 and S_2 to the *same* segment $[(0,0),(0,\rho)] \subset \mathbb{R}^2$ of length ρ. More precisely, we have

$\varphi_1(S_1) = \varphi_2(S_2) = [(0,0),(0,\rho)]$, with $\varphi_1(P) = \varphi_2(P) = (0,0)$ and $\varphi_1(P') = \varphi_2(P') = (0,\rho)$. The boundary of the surface is called the slit, the two segments S_1, S_2 that compose the slit are called the sides of the slit, and their common endpoints P, P' are the endpoints of the slit.

Throughout this article we will be interested in *the* flat slit torus, denoted X, whose metric comes from a *square* of area 1 (i.e., $X\backslash\{S_1,S_2\}$ is isometric to an open subset of a flat square torus) and whose slit of length $0 < \rho < 1$ is *vertical*, as these assumptions make the computations much easier. We fix an orientation on X so that it makes sense to talk about the left and right-hand sides of the slit and the upper and lower endpoint of the slit. Informally, the surface X can be visualized with the following polygonal model. Take a square of area 1 and cut it open along a vertical open interval of length ρ, then glue the opposites edges of the square by translation. Finally, glue one segment to each side of the cut part, so that the final surface is a compact genus 1 surface with boundary homeomorphic to a circle. The metric on the surface then simply comes from the canonical Euclidean metric on the square.

Since the slit torus is homotopy equivalent to the wedge sum of two circles, we have $\pi_1(X) \cong \mathbb{F}_2$, where $\mathbb{F}_2 = \mathbb{F}(s,t) = \langle s,t\rangle$ is the free group on the two generators s and t. More importantly, we also have $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$. From now on we will identify the elements h of $H_1(X,\mathbb{Z})$ with the integer couples (m,n) in $\mathbb{Z}^2$.

The Farey sequence. — Our first goal is to compute explicitly the stable norm of any given homology class on X. We also want to describe the structure of the unit ball $\mathcal{B}$ of the stable norm of X, which is a convex set in $\mathbb{R}^2$; more precisely, we want to know whether there are any segments contained in its boundary $\partial\mathcal{B}$, what its extreme points are, if there are any vertices, etc. It turns out that the stable norm of an integral homology class $(m,n) \in H_1(X,\mathbb{Z})$ depends on arithmetic properties of the rational number n/m. More precisely, it depends on its position in the Farey sequence.

The *Farey sequence of order* $k \in \mathbb{N}^*$ is the ordered sequence $\mathcal{F}_k$ of irreducible fractions in $\mathbb{Q}$ whose denominator is less than or equal to k. For example,

$$\mathcal{F}_1 = \left\{\dots,\frac{-1}{1},\frac{0}{1},\frac{1}{1},\frac{2}{1},\dots\right\} \text{ and } \mathcal{F}_2 = \left\{\dots,\frac{-1}{1},\frac{-1}{2},\frac{0}{1},\frac{1}{2},\frac{1}{1},\frac{3}{2},\frac{2}{1},\dots\right\}.$$

The Farey sequence is omnipresent in mathematics, mainly due to its remarkable combinatorial properties. Here are the main properties that we will need, which can be found in [12]. Let $a/b < c/d$ be two irreducible fractions that are consecutive in some Farey sequence $\mathcal{F}_k$ of order k. These fractions are called a Farey pair and are said to be Farey neighbours. They satisfy the following properties:

1. $bc - ad = 1$. Note that this property is actually equivalent to being Farey neighbours.

2. If p/q is an irreducible fraction such that a/b, p/q, and c/d are consecutive in some Farey sequence, then
$$\frac{p}{q} = \frac{a+c}{b+d}.$$
We denote by $p/q = a/b \oplus c/d$ and we say that p/q is the mediant of a/b and c/d.

Thus if a/b and c/d are Farey neighbours, the first fraction p/q that appears between them as the order of the sequence increases is their mediant; we say that p/q is the *Farey child* of a/b and c/d, and that a/b and c/d are the *Farey parents* of p/q.

3. Let $p/q = a/b \oplus c/d$ be an irreducible fraction, with $a/b < c/d$ its Farey parents. The Farey parents are the best rational approximations of p/q with denominator less than q. More precisely, $a/b < p/q < c/d$, and there are no other rational number than p/q with denominator less than q in the interval $[a/b, c/d]$. In other words, the last two convergents in the continued fraction expansion of p/q are its Farey parents.

Note that if a/b and c/d are the Farey parents of p/q, then the mediant $a/b \oplus p/q$ is also a Farey child of a/b. In fact, the children of a/b are all the rationals one of whose Farey parents is a/b; this is of great importance for our main result. Interestingly, the integers are the Farey ancestors of all the rationals numbers; indeed, starting from the integers $n/1$ and taking the successive mediants one can obtain any rational number.

Main result. — The main result of this paper is the explicit computation of the stable norm of the slit torus X, along with a complete description of the unit ball of the stable norm of X.

THEOREM A. — *Let X be the square flat torus with a vertical slit of length ρ. Let $L = \lfloor 1/\rho \rfloor$ be the integral part of $1/\rho$. Let $h = (m,n) \in H_1(X,\mathbb{Z})$ be a primitive integral homology class and let a/b and c/d be the Farey parents of n/m, with $a/b < c/d$. The unit ball $\mathcal{B}$ of the stable norm of X has a vertex in the direction h if and only if either*

- $\dfrac{n}{m} \in \mathcal{F}_L$, *that is to say* $1 \le |m| \le L$. *In that case*
$$\|(m,n)\|_X = \sqrt{m^2+n^2},$$

or

- $\dfrac{n}{m}$ *does not belong to* $\mathcal{F}_L$, *but* $\dfrac{a}{b} \in \mathcal{F}_L$ *or* $\dfrac{c}{d} \in \mathcal{F}_L$. *In that case, we have*
$$\|(m,n)\|_X = \sqrt{b^2+(a+\rho)^2} + \sqrt{d^2+(c-\rho)^2}.$$

Moreover, the stable norm is strictly convex in the directions $\pm(0,1)$, *and* $\|(0,1)\|_X = 1$. *Finally, the point of the unit sphere* $\partial\mathcal{B}$ *of the stable norm* *in every other direction of* $H_1(X,\mathbb{R})$ *lies in the interior of a segment. More precisely, if* $|m|,|b|,|d|>L$*, we have* $\|(m,n)\|_X=\|(b,a)\|_X+\|(d,c)\|_X$.

Note that since the unit ball $\mathcal{B}$ of the stable norm has infinitely many vertices it is not a polygon, although it very much looks like one at first glance. Indeed, the angles at the vertices are so close to being flat that $\mathcal{B}$ only seems to have a finite number of vertices. Sections 1 to 5 are devoted to proving this theorem.

Stable norm of a half-translation surface. — A question one might ask is which convex bodies can be realized as the unit ball of the stable norm of a manifold. In dimension $n\geq 3$, Babenko and Balacheff [3] showed that given a closed smooth manifold M, any centrally symmetric polytope with vertices in rational direction can be realized as the unit ball of the stable norm of a Riemannian metric on M. However, in dimension 2, Massart showed in [16] that the dimension of the flats of the unit ball of a closed orientable *smooth* surface of genus g is at most $g-1$; in particular, the unit ball of the stable norm is far from being a polytope. But what about the stable norm of singular metrics? In Section 6, we glue two slit tori X_i, $i=1,2$, along a long flat cylinder to obtain a genus 2 half-translation surface Σ on which we are able to compute the stable norm thanks to Theorem A.

THEOREM B. — *The unit ball of the stable norm of* Σ *is the convex hull of the set*

$$\left\{\frac{(m,n,0,0)}{\|(m,n,0,0\|_\Sigma}\ \textit{with}\ (m,n)\in V(X_1)\right\}\cup\left\{\frac{(0,0,p,q)}{\|(0,0,p,q)\|_\Sigma}\ \textit{with}\ (p,q)\in V(X_2)\right\},$$

where $V(X_i)$ *is the set of directions of vertices of the stable norm of* X_i.

In particular, the unit ball of the stable norm of Σ has faces of dimension 3, which is maximal since $H_1(\Sigma,\mathbb{R})$ has dimension 4. We will then use a similar construction to obtain a half-translation surface of genus g whose flats of the unit ball of its stable norm are of maximal dimension, that is $2g-1$. Again, the unit ball of the stable norm of Σ is not a polytope as it has infinitely many vertices, but it looks a lot like one.

Asymptotic growth of the number of simple homology classes. — The classical simple curve counting problem on surfaces is the following: how many *simple* closed curves of length less than a positive number x on a given surface are there? This question, along with many related problems, has seen significant progress during the last two decades. Now, one might ask which homology classes these curves represent. We say that an integral homology class on a (possibly singular) Riemannian surface S is *simple* if its stable norm is realized by the length of a simple closed curve. We then ask the following question: how many are there simple homology classes of stable norm less than x on S?

In dimension 2, it is known (see Balacheff–Massart [4]) that it is possible to find closed non-orientable surfaces with only a finite number of simple homology