

quatrième série - tome 45 fascicule 2 mars-avril 2012

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

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*Local energy decay for several evolution equations
on asymptotically Euclidean manifolds*

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LOCAL ENERGY DECAY FOR SEVERAL EVOLUTION EQUATIONS ON ASYMPTOTICALLY EUCLIDEAN MANIFOLDS

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ABSTRACT. – Let P be a long range metric perturbation of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geq 2$. We prove local energy decay for the solutions of the wave, Klein-Gordon and Schrödinger equations associated to P . The problem is decomposed in a low and high frequency analysis. For the high energy part, we assume a non trapping condition. For low (resp. high) frequencies we obtain a general result about the local energy decay for the group $e^{itf(P)}$ where f has a suitable development at zero (resp. infinity).

RÉSUMÉ. – Soit P une perturbation métrique à longue portée du laplacien euclidien sur $\mathbb{R}^d$, $d \geq 2$. On montre la décroissance de l'énergie locale des solutions des équations des ondes, de Klein-Gordon et de Schrödinger associées à P . Le problème est décomposé en une analyse basses et hautes fréquences. Afin de traiter les hautes fréquences, on fait une hypothèse de non capture. Pour les basses (resp. hautes) fréquences, on obtient un résultat général sur la décroissance de l'énergie locale pour le groupe $e^{itf(P)}$ où f a un comportement prescrit en zéro (resp. à l'infini).

1. Introduction

This paper is devoted to the study of the local energy decay for several evolution equations associated to long range metric perturbations of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geq 2$. In particular, we show that the local energy for the wave (resp. Schrödinger) equation decays like $\langle t \rangle^{1-d+\varepsilon}$ (resp. $|t|^{-d/2} \langle t \rangle^\varepsilon$). The restriction on the decays comes from the low frequency part, whose study constitutes the main part of the paper.

In the case of compactly supported (or exponentially decaying) perturbations, one of the most efficient approaches to prove local energy decay is the theory of resonances. For the wave equation outside some non trapping obstacles in odd dimension ≥ 3 , Lax and Phillips [25] have obtained an exponential decay of the local energy. This has been generalized by Lax, Morawetz and Phillips [24] to star-shaped obstacles and by Melrose and Sjöstrand [26] to non trapping obstacles. For general, non trapping, differential operators, Vainberg [38] has obtained exponential (resp. polynomial) decay in odd (resp. even) dimensions. His proof rests on estimates of the cut-off resolvent in the complex plane. The theory of resonances can also

be used to analyze the trapping situation; but, in this case, Ralston [29] has proved that there is necessarily a loss of derivatives in the estimates. We mention the work of Burq [10] without assumption on the trapped set, the work of Tang and Zworski [34] for the resonances close to the real line, the work of Christianson [11] for hyperbolic trapped sets with small topological pressure studied by Nonnenmacher and Zworski [27] and the work of Petkov and Stoyanov [28] outside several disjoint convex compact obstacles.

For slowly decaying perturbations, it is not clear how to use the theory of resonances. Instead, one can apply other methods (like resolvent estimates, perturbation theory, vector field methods, ...) giving typically polynomial decays. Jensen, Mourre and Perry [23] and Hunziker, Sigal and Soffer [21] have proved abstract local energy decays using Mourre theory. There is also a huge literature concerning the local energy decay for the Schrödinger equation perturbed by a potential. We only mention here the works of Rauch [30] and of Jensen and Kato [22]. Perturbation theory can also be applied for short range metric perturbations as in the work of Wang [41]. Schlag, Soffer and Staubach [32, 33] (see also the references therein) have considered radial short range perturbations of conical ends.

There has been important progress concerning the local energy decay for the wave equation in black hole type space-times. Finster, Kamran, Smoller and Yau [17], Tataru and Tohaneanu [36], Dafermos and Rodnianski [12], Andersson and Blue [2] and Tataru [35] have proved various results in this direction for the Kerr metric which is, far away from the black hole, a long range perturbation of the Minkowski metric.

In dimension 3, Tataru [35] has obtained a $\langle t \rangle^{-3}$ local decay rate for some wave equations with long range perturbations which are radial up to short range terms. In our long range setting, Bouclet [8] has established estimates for various evolution equations with other decay rates. Note that he also obtained low frequency estimates for powers of the resolvent (see also Bouclet [9] and our work [5] for estimates on the resolvent at low energy and Guillarmou and Hassell [19, 20] for a low frequency description of the resolvent using pseudodifferential calculus).

One can also consider evolution equations of higher order. For example, Ben-Artzi, Koch and Saut [4] have established different dispersive estimates for the fourth order Schrödinger groups $e^{it(\varepsilon\Delta+\Delta^2)}$ with $\varepsilon \in \{-1, 0, 1\}$. Moreover, Balabane [3] has obtained smoothing effects and local energy decays for evolution equations associated to an elliptic Fourier multiplier.

We consider the following operator on $\mathbb{R}^d$, with $d \geq 2$,

$$(1) \quad P = -b \operatorname{div}(G \nabla b) = - \sum_{i,j=1}^d b(x) \frac{\partial}{\partial x_i} G_{i,j}(x) \frac{\partial}{\partial x_j} b(x),$$

where $b(x) \in C^\infty(\mathbb{R}^d)$ and $G(x) \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ is a real symmetric $d \times d$ matrix. The C^∞ hypothesis is made mostly for convenience, much weaker regularity could actually be considered. We make an ellipticity assumption:

$$(H1) \quad \exists C > 0, \forall x \in \mathbb{R}^d \quad G(x) \geq CI_d \text{ and } b(x) \geq C,$$

I_d being the identity matrix on $\mathbb{R}^d$. We also assume that P is a long range perturbation of the Euclidean Laplacian:

$$(H2) \quad \exists \rho > 0, \forall \alpha \in \mathbb{N}^d \quad |\partial_x^\alpha (G(x) - I_d)| + |\partial_x^\alpha (b(x) - 1)| \lesssim \langle x \rangle^{-\rho - |\alpha|}.$$

With these assumptions, $(P, H^2(\mathbb{R}^d))$ is self-adjoint on $L^2(\mathbb{R}^d)$, $\sigma(P) = [0, +\infty[$ and P has no eigenvalues. The absence of positive eigenvalues follows from Donnelly [15, Corollary 5.4]. The absence of the zero eigenvalue is a consequence of the ellipticity Hypothesis (H1).

In particular, if $b = 1$, we are concerned with an elliptic operator in divergence form $P = -\operatorname{div}(G\nabla)$. On the other hand, if $G = (g^2 g^{i,j}(x))_{i,j}$, $b = (\det g^{i,j})^{1/4}$, $g = \frac{1}{b}$, then the above operator is unitarily equivalent to the Laplace-Beltrami $-\Delta_{\mathfrak{g}}$ on $(\mathbb{R}^d, \mathfrak{g})$ with metric

$$\mathfrak{g} = \sum_{i,j=1}^d g_{i,j}(x) dx^i dx^j,$$

where $(g_{i,j})_{i,j}$ is inverse to $(g^{i,j})_{i,j}$ and the unitary transform is just multiplication by g . In the following, $\|\cdot\|$ will always refer to the norm on $L^2(\mathbb{R}^d)$. We first obtain local energy decay estimates for several evolution equations at low frequency.

THEOREM 1. – Assume (H1)–(H2) and $d \geq 2$. For all $\chi \in C_0^\infty(\mathbb{R})$ and $\varepsilon > 0$, we have
i) for the wave equation

$$(2) \quad \left\| \langle x \rangle^{1-d} \frac{\sin t\sqrt{P}}{\sqrt{P}} \chi(P) \langle x \rangle^{1-d} \right\| \lesssim \langle t \rangle^{1-d+\varepsilon},$$

$$(3) \quad \left\| \langle x \rangle^{-d} (\partial_t, \sqrt{P}) \frac{\sin t\sqrt{P}}{\sqrt{P}} \chi(P) \langle x \rangle^{-d} \right\| \lesssim \langle t \rangle^{-d+\varepsilon}.$$

ii) for the Klein-Gordon equation

$$(4) \quad \left\| \langle x \rangle^{-d/2} e^{it\sqrt{1+P}} \chi(P) \langle x \rangle^{-d/2} \right\| \lesssim \langle t \rangle^{-d/2+\varepsilon}.$$

iii) for the Schrödinger equation

$$(5) \quad \left\| \langle x \rangle^{-d/2} e^{itP} \chi(P) \langle x \rangle^{-d/2} \right\| \lesssim \langle t \rangle^{-d/2+\varepsilon}.$$

iv) for the fourth order Schrödinger equation

$$(6) \quad \left\| \langle x \rangle^{-d/2} e^{it(P+P^2)} \chi(P) \langle x \rangle^{-d/2} \right\| \lesssim \langle t \rangle^{-d/2+\varepsilon},$$

$$(7) \quad \left\| \langle x \rangle^{-d/2} e^{itP^2} \chi(P) \langle x \rangle^{-d/2} \right\| \lesssim \langle t \rangle^{-d/4+\varepsilon}.$$

The above theorem collects special cases of a more general theorem.

THEOREM 2. – Assume (H1)–(H2) and $d \geq 2$. Let f be a real function such that

$$f(x) = a_0 + a_1 x^\alpha + x^{\alpha+\nu} g(x),$$

with $a_1 \neq 0$, $\alpha, \nu > 0$ and $g \in C^\infty(\mathbb{R})$. Let $\chi \in C_0^\infty(\mathbb{R})$ be such that $f'(x) > 0$ for all $x \in \operatorname{supp} \chi \cap]0, +\infty[$.

i) If $0 < \alpha \leq 1$, we have for all $\varepsilon > 0$

$$(8) \quad \left\| \langle x \rangle^{-\frac{d}{2\alpha}} e^{itf(P)} \chi(P) \langle x \rangle^{-\frac{d}{2\alpha}} \right\| \lesssim \langle t \rangle^{-\frac{d}{2\alpha}+\varepsilon}.$$

ii) If $\alpha > 1$, we have for all $\varepsilon > 0$

$$(9) \quad \left\| \langle x \rangle^{-\frac{d}{2}} e^{itf(P)} \chi(P) \langle x \rangle^{-\frac{d}{2}} \right\| \lesssim \langle t \rangle^{-\frac{d}{2\alpha}+\varepsilon}.$$

We now give estimates which are global in energy. Since the Hamiltonian trajectories play a crucial role at high frequencies, the local energy decay depends on the geometry of these curves. In this paper, we will assume that

(H3) P is non-trapping.

Under this assumption, we obtain local energy decay estimates for various evolution equations.

THEOREM 3. – Assume (H1)–(H3) and $d \geq 2$. For all $\varepsilon > 0$, we have
i) for the wave equation

$$(10) \quad \left\| \langle x \rangle^{1-d} \frac{\sin t\sqrt{P}}{\sqrt{P}} u \right\|_{H^1(\mathbb{R}^d)} \lesssim \langle t \rangle^{1-d+\varepsilon} \|\langle x \rangle^{d-1} u\|,$$

$$(11) \quad \left\| \langle x \rangle^{-d} (\partial_t, \sqrt{P}) \frac{\sin t\sqrt{P}}{\sqrt{P}} u \right\| \lesssim \langle t \rangle^{-d+\varepsilon} \|\langle x \rangle^d u\|.$$

ii) for the Klein-Gordon equation

$$(12) \quad \|\langle x \rangle^{-d/2} e^{it\sqrt{1+P}} u\| \lesssim \langle t \rangle^{-d/2+\varepsilon} \|\langle x \rangle^{d/2} u\|.$$

iii) for the Schrödinger equation

$$(13) \quad \|\langle x \rangle^{-d/2} e^{itP} u\| \lesssim |t|^{-d/2} \langle t \rangle^\varepsilon \|\langle x \rangle^{d/2} u\|_{H^{-d/2}(\mathbb{R}^d)}.$$

iv) for the fourth order Schrödinger equation

$$(14) \quad \|\langle x \rangle^{-d/2} e^{it(P+P^2)} u\| \lesssim |t|^{-d/2} \langle t \rangle^\varepsilon \|\langle x \rangle^{d/2} u\|_{H^{-3d/2}(\mathbb{R}^d)},$$

$$(15) \quad \|\langle x \rangle^{-d/2} e^{itP^2} u\| \lesssim |t|^{-d/2} \langle t \rangle^{d/4+\varepsilon} \|\langle x \rangle^{d/2} u\|_{H^{-3d/2}(\mathbb{R}^d)}.$$

REMARK 4. – i) In even dimensions, the decay rates in (10) and (11) are optimal modulo the loss of $\langle t \rangle^\varepsilon$. Indeed, the fundamental solution of the wave equation on the Minkowski space is explicitly known and a better estimate is not possible (see e.g. Taylor [37, Section 3.5]). In odd dimensions, better decay rates for the wave equation can be obtained for short range perturbations (see [7]).

ii) The type of decay we obtain for the wave equation does not depend on the parity of the dimension. This is not the case on the Minkowski space since the strong Huygens principle assures that the local energy decays as much as we want for $d \geq 3$ odd. For compactly supported perturbations, the theory of resonances gives an exponential decay (see e.g. Vainberg [38]). The difference with our results is that, roughly speaking, we only use upper bounds on the kernel of the resolvent (which do not depend on the parity of the dimension) and not analytic properties (only valid in odd dimensions).

iii) Note that the negative powers of $\langle x \rangle$ in the above theorem are necessary to obtain the required decays in time, modulo $\langle t \rangle^\varepsilon$. Indeed, weaker weights in $\langle x \rangle$ will give weaker decay rates. In this situation, the decay rates can be computed by interpolation using the estimates in the theorem.

iv) The decays that we obtain globally are limited by the best possible decays for the low frequency part given by Theorem 1. But, outside of the low frequencies, better decays follow from the estimates (16) and (17) below which hold in all dimension $d \geq 1$.