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CELLULAR E_k -ALGEBRAS

S. Galatius, A. Kupers & O. Randal-Williams

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CELLULAR E_k -ALGEBRAS

by

S. Galatius, A. Kupers & O. Randal-Williams

Abstract. – We give a set of foundations for cellular E_k -algebras which are especially convenient for applications to homological stability. We provide conceptual and computational tools in this setting, such as filtrations, a homology theory for E_k -algebras with a Hurewicz theorem, CW approximations, and many spectral sequences, which shall be used for such applications in future papers.

Résumé (E_k -algèbres cellulaires). – Nous bâtissons les fondations de la théorie des E_k -algèbres cellulaires, en vue d'applications en stabilité homologique. Nous donnons des outils conceptuels et des outils de calcul, y compris des filtrations, une théorie d'homologie pour les E_k -algèbres avec un théorème de Hurewicz, des approximations cellulaires, et plusieurs suites spectrales.

CONTENTS

1. Introduction	1
1.1. E_k -algebras	2
1.2. Cellular E_k -algebras	3
1.3. E_k -homology	4
1.4. Computing E_k -homology	5
1.5. Towards applications	6
1.6. Guide for the reader	7
Acknowledgments	8
 Part I. Category theory of algebras over a monad	 9
2. Contexts for category theory	11
2.1. Axioms for categories	11
2.2. Examples	15
2.3. Module categories	18
2.4. Diagram categories	19
3. Sifted monads and indecomposables	23
3.1. Monads and adjunctions	23
3.2. Sifted monads	24
3.3. The basepoint monad	27
3.4. Morphisms of monads and indecomposables	29
4. Monads associated to operads	33
4.1. Symmetric, braided and ordered sequences	33
4.2. Operads	37
4.3. Operads and lax monoidal functors	38
4.4. Non-unitary operads and unitalization	40
4.5. Augmentations of operads	41
4.6. Operads in simplicial sets	44
5. Filtered algebras	47
5.1. Graded and filtered objects	47
5.2. Functors between filtered and unfiltered objects	48
5.3. Monoidality	49
5.4. The canonical multiplicative filtration	53

6. Cell attachments	57
6.1. Cell attachments for sifted monads	57
6.2. Ascending filtrations from cell attachments	59
6.3. Cellular algebras and CW-algebras	65
Part II. Homotopy theory of algebras over a monad	69
7. Contexts for homotopy theory	71
7.1. Axioms for convenient contexts	71
7.2. Examples	73
7.3. Model category structures transferred along adjunctions	75
8. Homotopy theory of monads and indecomposables	81
8.1. Homotopy theory of algebras over a monad	81
8.2. Deriving constructions on T -algebras	83
8.3. Simplicial formulae	84
9. Homotopy theory of operads, algebras, and modules	91
9.1. Symmetric sequences	91
9.2. Algebras over operads	92
9.3. Simplicial formulae revisited	99
9.4. Modules over associative algebras	101
10. Homology and spectral sequences	105
10.1. Homology	105
10.2. Homotopy theory of filtered objects	112
10.3. Multiplicative filtrations of $\mathcal{O}$ -algebras	118
10.4. Filtrations of associative algebras and their modules	124
11. Hurewicz theorems and CW approximation	125
11.1. Connectivity functors	125
11.2. Hurewicz theorems for relative indecomposables	128
11.3. Hurewicz theorems for absolute indecomposables	131
11.4. Whitehead theorems	134
11.5. CW approximation in the semistable case	135
Part III. E_k-algebras	141
12. E_k-algebras and E_1-modules	143
12.1. E_k - and E_k^+ -operads	143
12.2. Modules over E_1 -algebras	147
13. Indecomposables and the bar construction	161
13.1. The iterated bar construction	161
13.2. The bar construction for modules	168
13.3. The bar construction on free algebras	168
13.4. Proof of Theorem 13.7	178
13.5. The bar construction on maps between free algebras	180

13.6. Iterated indecomposables	186
13.7. The E_∞ -case and infinite bar spectra	189
13.8. Group completion	191
14. Transferring vanishing lines	201
14.1. The bar spectral sequence	201
14.2. Transferring vanishing lines up	205
14.3. Transferring vanishing lines down	207
15. Comparing algebra and module cells	209
15.1. Coproducts of E_k -algebras	209
15.2. Comparison in the E_∞ case	212
15.3. Comparison in general	215
15.4. Proof of Theorem 15.3	218
16. W_{k-1}-algebras	229
16.1. Homology operations on E_k -algebras	229
16.2. Relations among homology operations	231
16.3. The homology of free E_k -algebras	237
16.4. Coproducts of E_∞ -algebras	238
16.5. E^1 -pages of spectral sequences	238
16.6. W_{k-1} -algebra structures on spectral sequences	239
16.7. Derived $\tilde{W}_{k-1}$ -indecomposables	241
Part IV. A framework for examples	243
17. E_k-algebras from monoidal groupoids	245
17.1. Constructing E_k -algebras	245
17.2. E_1 -splitting complexes	246
17.3. E_k -splitting complexes	250
17.4. E_∞ -splitting complexes	252
17.5. E_1 -splitting complexes revisited	258
18. Application to homological stability	259
18.1. A generic homological stability result	259
18.2. Example: general linear groups of Dedekind domains	265
18.3. Example: general linear groups of $\mathbb{F}_q$	266
19. Local coefficients	269
19.1. Coefficient systems	269
19.2. Polynomial coefficients	273
19.3. Example: general linear groups of $\mathbb{F}_q$ with local coefficients	277
20. Koszulity and connectivity	279
20.1. Quadratic data	279
20.2. Duality and the E_1 -homology of quadratic algebras	280
20.3. The fundamental example	282

Bibliography	285
List of Symbols	293
Index	297