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CREATION OF FERMIONS BY ROTATING CHARGED BLACK HOLES

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CREATION OF FERMIONS BY ROTATING CHARGED BLACK HOLES

Dietrich HÄFNER

Abstract. – This work is devoted to the mathematical study of the Hawking effect for fermions in the setting of the collapse of a rotating charged star. We show that an observer who is located far away from the star and at rest with respect to the Boyer Lindquist coordinates observes the emergence of a thermal state when his proper time goes to infinity. We first introduce a model of the collapse of the star. We suppose that the space-time outside the star is given by the Kerr-Newman metric. The assumptions on the asymptotic behavior of the surface of the star are inspired by the asymptotic behavior of certain timelike geodesics in the Kerr-Newman metric. The Dirac equation is then written using coordinates and a Newman-Penrose tetrad which are adapted to the collapse. This coordinate system and tetrad are based on the so called simple null geodesics. The quantization of Dirac fields in a globally hyperbolic space-time is described. We formulate and prove a theorem about the Hawking effect in this setting. The proof of the theorem contains a minimal velocity estimate for Dirac fields that is slightly stronger than the usual ones and an existence and uniqueness result for solutions of a characteristic Cauchy problem for Dirac fields in the Kerr-Newman space-time. In an appendix we construct explicitly a Penrose compactification of block I of the Kerr-Newman space-time based on simple null geodesics.

Résumé **(Création de fermions par des trous noirs chargés en rotation)**

Ce travail est consacré à l'étude mathématique de l'effet Hawking pour des fermions dans le cadre de l'effondrement d'une étoile chargée en rotation. On démontre qu'un observateur localisé loin de l'étoile et au repos par rapport aux coordonnées de Boyer-Lindquist observe l'émergence d'un état thermal quand son temps propre tend vers l'infini. On introduit d'abord un modèle de l'effondrement de l'étoile. On suppose que l'espace-temps à l'extérieur de l'étoile est donné par la métrique de Kerr-Newman. Les hypothèses sur le comportement asymptotique de la surface de l'étoile sont inspirées par le comportement asymptotique de certaines géodésiques de type temps dans la métrique de Kerr-Newman. L'équation de Dirac est alors écrite en utilisant des coordonnées et une tétrade de Newman-Penrose adaptées à l'effondrement. Ce système de coordonnées et cette tétrade sont basés sur des géodésiques qu'on appelle des géodésiques simples isotropes. La quantification des champs de Dirac dans un espace-temps globalement hyperbolique est décrite. On formule un théorème sur l'effet Hawking dans ce cadre. La preuve du théorème contient une estimation de vitesse minimale pour les champs de Dirac légèrement plus forte que les estimations usuelles ainsi qu'un résultat d'existence et d'unicité pour les solutions d'un problème caractéristique pour les champs de Dirac dans l'espace-temps de Kerr-Newman. Dans un appendice, nous construisons explicitement la compactification de Penrose du bloc I de l'espace-temps de Kerr-Newman qui est basée sur les géodésiques simples isotropes.

CONTENTS

CHAPTER 1

INTRODUCTION

It was in 1975 that S.W. Hawking published his famous paper about the creation of particles by black holes (see [**30**]). Later this effect was analyzed by other authors in more detail (see e.g. [**47**]) and we can say that the effect was well understood from a physical point of view at the end of the 1970's. From a mathematical point of view, however, fundamental questions linked to the Hawking radiation such as scattering theory for field equations on black hole space-times had not been addressed at that time.

In the early 1980's Dimock and Kay started a research programme concerning scattering theory on curved space-times. They obtained an asymptotic completeness result for classical and quantum massless scalar fields on the Schwarzschild metric (see [**21**]–[**20**]). Their work was pushed further by Alain Bachelot in the 1990's. He showed asymptotic completeness for Maxwell and Klein-Gordon fields (see [**1**], [**2**]) and gave a mathematically precise description of the Hawking effect (see [**3**]–[**5**]) in the spherically symmetric case. Meanwhile other authors contributed to the subject such as Nicolas [**38**], Jin [**33**] and Melnyk [**36**], [**37**]. All these works deal with the spherically symmetric case.

The more realistic case of a rotating black hole is more difficult. In the spherically symmetric case, the study of a field equation can be reduced to the study of a $1+1$ dimensional equation with potential. In the Kerr case this reduction is no longer possible and the methods used in the papers cited so far do not apply. A paper by De Bièvre, Hislop, Sigal using different methods appeared in 1992 (see [**10**]). By means of a Mourre estimate they show asymptotic completeness for the wave equation on non-compact Riemannian manifolds; possible applications are therefore static situations such as the Schwarzschild case, which they treat, but the Kerr geometry is not even stationary. In this context we also mention the paper of Daudé about the Dirac equation in the Reissner-Nordström metric (see [**14**]). A complete scattering theory for the wave equation on stationary, asymptotically flat space-times, was obtained by the author in 2001 (see [**27**]). To our knowledge the first asymptotic completeness result in the Kerr case was obtained by the author in [**28**], for the non superradiant modes of the Klein-Gordon field. The first complete scattering theory for a field equation in the Kerr metric was obtained by Nicolas and the author in [**29**] for massless Dirac fields. This result was generalized by Daudé in [**15**] to the massive charged Dirac field in the Kerr-Newman metric. All these papers use Mourre theory.

The aim of the present paper is to give a mathematically precise description of the Hawking effect for spin $\frac{1}{2}$ fields in the setting of the collapse of a rotating charged star. We show that an observer who is located far away from the black hole and at rest with respect to the Boyer-Lindquist coordinates observes the emergence of a thermal state when his proper time t goes to infinity. Let us give an idea of the theorem describing the effect. Let r_* be the Regge-Wheeler coordinate. We suppose that the boundary of the star is described by $r_* = z(t, \theta)$. The space-time is then given by

$$\mathscr{M}_{\mathrm{col}} = \bigcup_t \Sigma_t^{\mathrm{col}}, \quad \Sigma_t^{\mathrm{col}} = \big\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2; r_* \geq z(t, \theta)\big\}.$$

The typical asymptotic behavior of $z(t, \theta)$ is ($\kappa_+ > 0$):

$$z(t, \theta) = -t - A(\theta)\,\mathrm{e}^{-2\kappa_+ t} + B(\theta) + \mathscr{O}(\mathrm{e}^{-4\kappa_+ t}), \quad t \to \infty.$$

Let $\mathscr{H}_t = L^2((\Sigma_t^{\mathrm{col}}, \mathrm{dVol}); \mathbb{C}^4)$. The Dirac equation can be written as

$$\partial_t \Psi = i\, \not{D}_t\, \Psi + \text{boundary condition}. \tag{1.1}$$

We will put a MIT boundary condition on the surface of the star. The evolution of the Dirac field is then described by an isometric propagator $U(t, s) : \mathscr{H}_s \to \mathscr{H}_t$. The Dirac equation on the whole exterior Kerr-Newman space-time $\mathscr{M}_{\mathrm{BH}}$ will be written as

$$\partial_t \Psi = i\, \not{D}\, \Psi.$$

Here $\not{D}$ is a selfadjoint operator on $\mathscr{H} = L^2((\mathbb{R}_{r_*} \times S^2, \mathrm{d}r_* \mathrm{d}\omega); \mathbb{C}^4)$. There exists an asymptotic velocity operator $P^\pm$ such that for all continuous functions J with $\lim_{|x|\to\infty} J(x) = 0$ we have

$$J(P^\pm) = s - \lim_{t \to \pm\infty} \mathrm{e}^{-it\not{D}} J\left(\frac{r_*}{t}\right) \mathrm{e}^{it\not{D}}.$$

Let $\mathscr{U}_{\mathrm{col}}(\mathscr{M}_{\mathrm{col}})$ (resp. $\mathscr{U}_{\mathrm{BH}}(\mathscr{M}_{\mathrm{BH}})$) be the algebras of observables outside the collapsing body (on the space-time describing the eternal black hole) generated by $\Psi^*_{\mathrm{col}}(\Phi_1)\Psi_{\mathrm{col}}(\Phi_2)$ (resp. $\Psi^*_{\mathrm{BH}}(\Phi_1)\Psi_{\mathrm{BH}}(\Phi_2)$). Here $\Psi_{\mathrm{col}}(\Phi)$ (resp. $\Psi_{\mathrm{BH}}(\Phi)$) are the quantum spin fields on $\mathscr{M}_{\mathrm{col}}$ (resp. $\mathscr{M}_{\mathrm{BH}}$). Let ω_{col} be a vacuum state on $\mathscr{U}_{\mathrm{col}}(\mathscr{M}_{\mathrm{col}})$; ω_{vac} a vacuum state on $\mathscr{U}_{\mathrm{BH}}(\mathscr{M}_{\mathrm{BH}})$ and $\omega^{\eta,\sigma}_{\mathrm{Haw}}$ be a KMS-state on $\mathscr{U}_{\mathrm{BH}}(\mathscr{M}_{\mathrm{BH}})$ with inverse temperature $\sigma > 0$ and chemical potential $\mu = \mathrm{e}^{\sigma\eta}$ (see Chapter 5 for details). For a function $\Phi \in C_0^\infty(\mathscr{M}_{\mathrm{BH}})$ we define

$$\Phi^T(t, r_*, \omega) = \Phi(t - T, r_*, \omega).$$

The theorem about the Hawking effect is:

Theorem 1.1 (Hawking effect). – *Let*

$$\Phi_j \in \big(C_0^\infty(\mathscr{M}_{\mathrm{col}})\big)^4, \quad j = 1, 2.$$