ISSN 0012-9593

quatrième série - tome 58 *fascicule 5* *septembre-octobre 2025*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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45, rue d'Ulm, 75230 Paris Cedex 05, France.
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Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 494 euros.
Abonnement avec supplément papier :
Europe : 694 €. Hors Europe : 781 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 58, 2025, p. 1089 à 1124

SUBLEADING ASYMPTOTIC OF LINK SPECTRAL INVARIANTS AND HOMEOMORPHISM GROUPS OF SURFACES

BY DAN CRISTOFARO-GARDINER, VINCENT HUMILIÈRE, CHEUK YU MAK, SOBHAN SEYFADDINI AND IVAN SMITH

ABSTRACT. – This paper continues the study of link spectral invariants on compact surfaces, introduced in our previous work and shown to satisfy a Weyl law in which they asymptotically recover the Calabi invariant. Here we study their subleading asymptotics on surfaces of genus zero. We show the subleading asymptotics are bounded for smooth time-dependent Hamiltonians, and recover the Ruelle invariant for autonomous disk maps with finitely many critical values. We deduce that the Calabi homomorphism admits infinitely many extensions to the group of compactly supported area-preserving homeomorphisms, and that the kernel of the Calabi homomorphism on the group of hameomorphisms is not simple.

RÉSUMÉ. – Cet article poursuit l'étude des invariants spectraux d'entrelacs introduits dans notre précédent travail, dans lequel il est établi qu'ils vérifient une loi de Weyl faisant apparaître l'invariant de Calabi asymptotiquement. Nous étudions ici leur asymptotique sous-dominante sur les surfaces de genre nul. Nous montrons que celle-ci est bornée pour tous les hamiltoniens lisses dépendant du temps, et qu'elle fait apparaître l'invariant de Ruelle pour les hamiltoniens autonomes du disque ayant un nombre fini de valeurs critiques. Nous en déduisons que le morphisme de Calabi admet une infinité d'extensions au groupe des homéomorphismes à support compact qui préservent l'aire, et que le noyau du morphisme de Calabi sur le groupe des haméomorphismes n'est pas simple.

1. Introduction

1.1. Area-preserving homeomorphisms of surfaces

Let (M, ω) be a compact manifold possibly with boundary, equipped with a volume-form, and consider the group $\mathrm{Homeo}_c(M, \omega)$ of volume-preserving homeomorphisms that are the identity near the boundary, in the component of the identity.

When the dimension of M is at least three, there is a clear picture due to Fathi regarding the algebraic structure of this group: there is a mass-flow homomorphism, and its kernel is a simple group. In contrast, in dimension two the situation is much less understood despite the fact that many decades have passed since Fathi's work. We refer the reader to

doi:10.24033/asens.2625

[8], e.g., Thm. 1.3, for background, including a definition of the mass-flow homomorphism [8, Sec. 2.3]. In the cases that we will be mainly concerned with here, that is the disk and the sphere, the kernel of mass-flow is just the group of area and orientation preserving homeomorphisms.

We recently showed [8, Thm. 1.3] that when $\dim(M) = 2$, the kernel of mass-flow is never simple. In fact, it contains as a proper normal subgroup the group $\mathrm{Hameo}(M, \omega)$ of *hameomorphisms*, whose definition we review in Definition 2.2. When M has boundary, we also showed that the classical Calabi homomorphism, which we review in Definition 2.1 and which measures the average rotation of the map, extends to Hameo from the group $\mathrm{Ham}_c(M, \omega)$ of Hamiltonian diffeomorphisms that are the identity near the boundary. It is then natural to ask the following.

QUESTION 1.1. – *When M is closed, is* $\mathrm{Hameo}(M, \omega)$ *simple? When M has non-empty boundary, is the kernel of Calabi on* $\mathrm{Hameo}(M, \omega)$ *simple?*

This is an old question. For example, a variant appears in [27, Problem (4)]. Let us briefly explain why one might hope for a positive answer. Hameomorphisms are homeomorphisms with well-defined Hamiltonians, and it is natural to wonder whether the algebraic structure of the group of hameomorphisms could be like that of the group $\mathrm{Ham}_c(M, \omega)$; moreover, Banyaga showed [1] that Ham_c is simple when M is closed and the kernel of Calabi is simple when M has boundary.

Our first result shows that the structure of Hameo is more complicated than this.

THEOREM 1.2. – *The following groups are not perfect:*

1. *The kernel of Calabi on* $\mathrm{Hameo}(\mathbb{D}^2, \omega)$.
2. *The group* $\mathrm{Hameo}(\mathbb{S}^2, \omega)$.

Both admit surjective group homomorphisms to $\mathbb{R}$.

Recall that a group G is called *perfect* if it coincides with its commutator subgroup $[G, G]$. Note that, since the commutator subgroup is always normal, every (non-abelian) simple group is perfect and hence we conclude that neither of the groups appearing in the above theorem are simple.

1.2. A two-term Weyl law

Theorem 1.2 is proved by studying the asymptotics of the "link spectral invariants" defined in our previous work [8, Thm. 1.13, Def. 6.14, Eq. (59)]. In [8, Sec. 7.3] we defined quasimorphisms

$$\mu_k : \mathrm{Diff}(\mathbb{S}^2, \omega) \to \mathbb{R}, \quad f_k : \mathrm{Homeo}_c(\mathbb{D}^2, \omega) \to \mathbb{R}$$

and we showed that these satisfy the important asymptotic formulae

$$\lim_{k \to \infty} f_k(g) = \mathrm{Cal}(g) \tag{1}$$

on $\mathrm{Diff}_c(\mathbb{D}^2, \omega)$, and

$$\lim_{k \to \infty} \mu_k(g) = 0.$$

We called this the "Calabi property". Here, Cal denotes the aforementioned Calabi homomorphism and Diff_c denotes the group of diffeomorphisms that are the identity near the boundary and that preserve ω, which we note for the reader coincides with the group Ham_c in the above cases. We refer the reader to our review in Section 2 for more details about the μ_k and f_k.

The above formulas are kinds of Weyl laws. For specialists, we note that the convergence to zero for the μ_k is what one would hope for in a Weyl law, since these invariants are defined via mean normalization of Hamiltonians. It is natural to ask what can be said about the subleading asymptotics. With many seemingly similar kinds of Weyl laws, this tends to be a hard question. For example, the above Calabi property was inspired by an analogous Weyl law for the related "ECH spectral invariants" defined in [19], see [11]. For these spectral invariants, all that is known is a bound on the growth rate of the subleading asymptotics [7] that is likely far from optimal, with the conjectural bound being $O(1)$ [20].

In contrast, it turns out that we are able to say quite a lot about the subleading asymptotics of the μ_k. To state our result, let Ru denote the Ruelle invariant from [32] (see also [17, 18]), which we review in Section 2.1.3. We now state a result that is central to our proof of Theorem 1.2 and which is also of independent interest.

Theorem 1.3. – *If $\psi \in \mathrm{Diff}(\mathbb{S}^2, \omega)$ (resp. $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega) \cap \ker(\mathrm{Cal})$), then the sequence $\{k\,\mu_k(\psi)\}_{k\in\mathbb{N}}$ (resp. $\{k\, f_k(\psi)\}_{k\in\mathbb{N}}$) is bounded. In fact, if $\psi = \phi^1_H$, where $H : \mathbb{D}^2 \to \mathbb{R}$ is an autonomous and compactly supported Hamiltonian on the disk with finitely many critical values, then*

$$\lim_k k\mu_k(\psi) = \lim_k k(f_k(\psi) - \mathrm{Cal}(\psi)) = \mathrm{Cal}(\psi) - \frac{1}{2}\mathrm{Ru}(\psi). \tag{2}$$

A similar result concerning the subleading asymptotics of the μ_k in the case of autonomous Hamiltonians on the sphere with finitely many critical values also holds, but for brevity (and because the Ruelle invariant is not defined over the sphere without further choices), we do not state it.

Remark 1.4. – In the statement of the above theorem, we are implicitly invoking the fact that we can regard any $\psi \in \mathrm{Diff}_c(\mathbb{D}^2, \omega)$ as a map of the two-sphere by embedding $\mathbb{D}^2$ as a hemisphere and extending by the identity, for our conventions see Section 2.2.4, when we write $\mu_k(\psi)$ in (2); we will continue to do this throughout this paper. The invariants μ_k and f_k can be thought of as invariants of (possibly time-dependent) Hamiltonians as well, by setting $\mu_k(H) := \mu_k(\phi^1_H)$ and $f_k(H) := f_k(\phi^1_H)$. This viewpoint is helpful and adopted in [8, Sec. 3], as well as Section 3 here.

In view of Theorem 1.3 it is natural to ask if (2) holds more generally. For the aforementioned ECH spectral invariants, essentially the same question was asked, under a genericity assumption on the contact form [20]. In the ECH case, simple examples exist, for example the boundary of the round sphere, with no well-defined subleading asymptotic limit at all; in this sense, then, the genericity assumption can not be dropped. In our case, however, we know of no such analog, and indeed Theorem 1.3 asserts that in the simplest cases, the subleading asymptotics in fact always recover Ruelle. We therefore pose as a question the following.