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Nessim SIBONY†, Andrey SOLDATENKOV & Misha VERBITSKY

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88.
Email : `annales@ens.fr`

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Société Mathématique de France
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RIGID CURRENTS ON COMPACT HYPERKÄHLER MANIFOLDS

BY NESSIM SIBONY†, ANDREY SOLDATENKOV AND MISHA VERBITSKY

ABSTRACT. – A rigid cohomology class on a complex manifold is a class that is represented by a unique closed positive current. The positive current representing a rigid class is also called rigid. On a compact Kähler manifold X, any eigenvector of a hyperbolic automorphism acting on $H^{1,1}(X)$ with a non-unit eigenvalue corresponds to a rigid class. Such classes are always parabolic, namely, they belong to the boundary of the Kähler cone and have vanishing volume. We study parabolic $(1,1)$-classes on compact hyperkähler manifolds with $b_2 \geqslant 7$. We show that a parabolic class is rigid if it is not orthogonal to a rational vector with respect to the BBF form. This implies that a very general parabolic class on a hyperkähler manifold is rigid.

RÉSUMÉ. – Une classe de cohomologie rigide sur une variété complexe est une classe représentée par un unique courant positif fermé. Un courant positif qui représente une classe rigide est dit rigide. Pour une variété kählerienne compacte X, tous les vecteurs propres d'automorphismes hyperboliques agissant sur $H^{1,1}(X)$ qui ont des valeurs propres de valeur absolue différente de un sont des classes rigides. Ces classes sont toujours paraboliques, c'est-à-dire qu'elles sont dans le bord du cône de Kähler et sont de volume nul. Nous étudions les $(1,1)$-classes paraboliques sur les variétés hyperkähleriennes compactes satisfaisant $b_2 \geqslant 7$. Nous montrons qu'une classe parabolique est rigide si elle n'est pas orthogonale à un vecteur rationnel pour la forme BBF. Cela implique qu'une classe parabolique générale sur une variété hyperkählerienne est rigide.

1. An introduction to rigid currents

1.1. Rigid currents in complex dynamics

Rigid currents on complex manifolds are positive closed (p,p)-currents that admit a unique positive closed representative in their cohomology class. More specifically, let X be a compact Kähler manifold of dimension k. Consider a cohomology class $\alpha \in H^{p,p}_{\mathbb{R}}(X)$, where the subscript $\mathbb{R}$ means that $\bar{\alpha} = \alpha$. The question we are interested in is whether it is

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possible to represent α by a closed positive current and to what extent such a representation is unique (see, e.g., [15] for the definition of positive currents and an overview of the underlying theory). Thinking of currents as differential forms whose coefficients are distributions, we denote the space of currents of Hodge type (p,q) by $\mathcal{E}'(X)^{p,q}$. For a current $T \in \mathcal{E}'(X)^{p,p}_{\mathbb{R}}$ we will write $T \geqslant 0$ and call T positive if for any $\eta_1, \ldots, \eta_{k-p} \in \bigwedge^{1,0} X$ we have

$$(\sqrt{-1})^{k-p}\langle T, \eta_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \eta_{k-p} \wedge \bar{\eta}_{k-p}\rangle \geqslant 0.$$

For a closed current T we denote by $[T]$ its cohomology class. Define

$$\mathcal{C}_\alpha = \{T \in \mathcal{E}'(X)^{p,p}_{\mathbb{R}} \mid T \geqslant 0,\ dT = 0,\ [T] = \alpha\}.$$

The class α is called *rigid* if $\mathcal{C}_\alpha$ contains exactly one element. The unique positive closed current representing a rigid class is also called rigid. Restricting to the case $p = 1$, recall that a class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ is called *pseudo-effective* if $\mathcal{C}_\alpha \neq \emptyset$. The pseudo-effective classes form a closed convex cone $\mathcal{P}_X \subset H^{1,1}_{\mathbb{R}}(X)$. A pseudo-effective class is called *nef* if it lies in the closure of the Kähler cone of X.

To put our work in context, let us recall how rigid currents appear in holomorphic dynamics, see, e.g., [19, 20]. For a holomorphic automorphism $f : X \to X$ the p-th dynamical degree $d_p(f)$ is defined as the spectral radius of f^* acting on $H^{p,p}(X)$ for $0 \leqslant p \leqslant k$, where $k = \dim_{\mathbb{C}}(X)$. It follows from [24] that the function $p \mapsto \log(d_p(f))$ is concave and the sequence of dynamical degrees is of the following form:

$$1 = d_0 < d_1 < \cdots < d_{p_0} = \cdots = d_{p_1} > \cdots > d_{k-1} > d_k = 1.$$

We will call a non-zero closed positive current $T \in \mathcal{E}'(X)^{p,p}_{\mathbb{R}}$ a *Green current* for f of order p if $f^*T = d_p(f)T$. Recall the following result which is a special case of [20, Theorem 4.2.1] combined with [20, Theorem 4.3.1].

THEOREM 1.1 (Dinh-Sibony, [20]). – *Let f be a holomorphic automorphism of a compact Kähler manifold X and $d_j(f)$ the dynamical degrees of f. Assume that $d_{p-1}(f) < d_p(f)$ for some $p \geqslant 1$. Then there exists a Green current T of order p for f and this current is rigid.*

The dynamical properties of automorphisms have been classically used to study the geometry and arithmetic of K3 surfaces, see, e.g., [41] and the works of Cantat [8, 9], who has discovered and studied dynamical currents on K3 surfaces. We refer to [20] for much more precise and general results of this form, and to the references in loc. cit. for an overview of how rigid currents arise in other settings.

Note that when the above theorem applies to $p = 1$ and T is the corresponding Green $(1,1)$-current, its cohomology class $\alpha = [T] \in H^{1,1}_{\mathbb{R}}(X)$ satisfies $\alpha^k = 0$, where as before $k = \dim_{\mathbb{C}}(X)$. This follows from the condition $d_1(f) > 1$ and the fact that the automorphism f acts as the identity on the top degree cohomology of X.

We introduce the following terminology: a cohomology class $\beta \in H^{1,1}_{\mathbb{R}}(X)$ is called *parabolic* if it is nef and $\beta^k = 0$.

1.2. Rigid currents on complex tori

We give an example of a rigid current which is smooth, unlike almost all rigid currents considered later in this paper. This subsection is included here to illustrate the concept of rigidity with the most elementary and hands-on example. However, the mapping class group argument (Subsection 5.5) which is central for our work, can be applied to a compact torus as well, bringing another proof of the following proposition.

PROPOSITION 1.2. – *Consider a translation-invariant holomorphic foliation $\mathcal{F} \subset TX$ on a compact complex torus X with* $\mathrm{rk}(\mathcal{F}) = d$. *Suppose that ω is a closed positive* $(1,1)$-*form vanishing on $\mathcal{F}$ and strictly positive in the transversal directions. Assume that one leaf of $\mathcal{F}$, and hence all leaves of $\mathcal{F}$, are dense in X. Then ω is rigid.*

Proof. – Denote by I the complex structure on X. Assume that $\omega_1 \in \mathcal{E}'(X)^{1,1}_{\mathbb{R}}$ is a closed positive current with $[\omega_1] = [\omega] \in H^2(X, \mathbb{R})$.

Step 1. – Given a vector field v tangent to $\mathcal{F}$, we claim that $i_v\omega_1 = 0$, where the contraction $i_v\omega_1$ is the $(0,1)$-current defined by duality: $\langle i_v\omega_1, \eta\rangle = \langle \omega_1, i_v\eta\rangle$ for any test-form η. Indeed, assume that $i_v\omega_1 \neq 0$ for some vector field v tangent to $\mathcal{F}$. Then $i_v i_{I(v)}\omega_1$ is a non-negative integrable function on X. Averaging v over the translations by the torus, we may assume that v is translation-invariant. Then $v \wedge Iv$ represents a homology class in $H_{1,1}(X, \mathbb{R})$ such that $\langle [\omega_1], [v \wedge Iv]\rangle > 0$. On the other hand, since ω vanishes on $\mathcal{F}$, we have $\langle [\omega], [v \wedge Iv]\rangle = 0$, contradicting the assumption $[\omega_1] = [\omega]$.

Step 2. – By the dd^c-lemma, $\omega_1 - \omega = dd^c f$ for some distribution f. Since $\omega_1 = \omega + dd^c f$ is positive, it is known (see, e.g., [26, Theorem A.5]) that f is represented by a quasi-plurisubharmonic function. By Step 1, for any vector field v tangent to $\mathcal{F}$, the derivative of f in the direction of v vanishes. Note that the restriction of f to any leaf of $\mathcal{F}$ can not be identically equal to $-\infty$: since the leaves are dense in X and f is quasi-plurisubharmonic, f would then be identically $-\infty$ everywhere on X. It follows that f is pluriharmonic along any leaf of $\mathcal{F}$. Since f is upper semicontinuous (see [25]), it attains its maximum at some point of X. By the maximum principle, f is constant on the leaf of $\mathcal{F}$ passing through that point. Since the leaves are dense in X, f is constant and $\omega_1 = \omega$. □

When the foliation $\mathcal{F}$ is obtained as the unstable foliation of an Anosov automorphism of a torus, 1.2 was already shown in [8, Remarque 2.3].

In the setting of complex surfaces, the rigid currents supported on 1-dimensional complex foliations (and even laminations) were studied extensively in [18], obtaining rigidity in a more general context.

1.3. Rigid currents on complex surfaces

Another simple example of a rigid pseudo-effective class can be constructed as follows. Assume that X is a Kähler surface. Then the pseudo-effective cone $\mathcal{P}_X$ is the dual of the nef cone, see, e.g., [7, Theorem 4.1]. Assume that $C \subset X$ is an irreducible curve with $C^2 < 0$. Then its cohomology class $[C]$ spans an extremal ray of $\mathcal{P}_X$, see, e.g., [7, Theorem 3.21]. The cohomology class $[C]$ is represented by the current of integration over C, and that is the unique closed positive current representing $[C]$, see [7, Proposition 3.15]. In this example the