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ALGEBRAIC VECTOR BUNDLES AND p-LOCAL $\mathbb{A}^1$-HOMOTOPY THEORY

BY Aravind ASOK, Jean FASEL AND Michael J. HOPKINS

Abstract. – Using techniques of $\mathbb{A}^1$-homotopy theory, we produce motivic lifts of elements in classical homotopy groups of spheres; these lifts provide polynomial maps of spheres and allow us to construct "low rank" algebraic vector bundles on "simple" smooth affine varieties of high dimension.

Résumé. – En utilisant des techniques d'homotopie des schémas, nous produisons des relèvements motiviques de certains éléments dans les groupes d'homotopie (instables) de sphères. Ces relèvements nous permettent de construire des fibrés vectoriels de "petit rang" sur des variétés algébriques « simples » de grande dimension, ainsi que de produire des représentants polynomiaux des éléments considérés.

1. Introduction

Fix a base field $k \subset \mathbb{C}$ and suppose X is a smooth algebraic k-variety. There is a forgetful map $\mathscr{V}_r(X) \to \mathscr{V}_r^{\mathrm{top}}(X)$ from the set of isomorphism classes of rank r algebraic vector bundles on X to the set of isomorphism classes of rank r complex topological vector bundles on the complex manifold $X(\mathbb{C})$ (throughout this paper, we abuse notation and write X for $X(\mathbb{C})$). A complex topological vector bundle lying in the image of this map is called *algebraizable*. In general, the forgetful map is neither injective nor surjective. A necessary condition for algebraizability of a vector bundle is that the topological Chern classes should be algebraic, i.e., should lie in the image of the cycle class map $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X, \mathbb{Z})$.

The forgetful map from the previous paragraph factors as:

$$\mathscr{V}_n(X) \longrightarrow [X, \mathrm{Gr}_n]_{\mathbb{A}^1} \longrightarrow \mathscr{V}_n^{\mathrm{top}}(X),$$

where $[X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ is an "algebraic" homotopy invariant mirroring classical homotopy invariance of topological vector bundles. In more detail, Gr_n is an infinite Grassmannian; it may be

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realized as the ind-scheme $\operatorname{colim}_N \mathrm{Gr}_{n,N}$ (see [27, §4 Proposition 3.7] for further discussion). The set $[X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ is the set of maps between X and Gr_n in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category [27].

The set $[X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ has a concrete description that we now give. F. Morel proved that if X is furthermore smooth and affine, then $[X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ coincides with the set of isomorphism classes of rank n vector bundles on X and also the quotient of the set of morphisms $X \to \mathrm{Gr}_n$ by the equivalence relation generated by $\mathbb{A}^1$-homotopies (see [9, Theorem 1]). For a smooth variety X, a result of Jouanolou-Thomason [37, Proposition 4.4] guarantees that there exists a smooth affine variety $\tilde{X}$ and a torsor under a vector bundle $\pi : \tilde{X} \to X$; by construction π is an isomorphism in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category (any such pair $(\tilde{X}, \pi)$ is called a Jouanolou device for X). Thus, any element of the set $[X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ may be represented by an equivalence class of morphisms $\tilde{X} \to \mathrm{Gr}_n$, i.e., by an actual vector bundle of rank n on $\tilde{X}$; we refer to such equivalence classes as *motivic vector bundles of rank n*.

In [7], we used the above factorization to demonstrate the existence of additional cohomological restrictions to algebraizability of a bundle beyond algebraicity of Chern classes. The obstructions we described relied on the failure of injectivity of the cycle class map. In this paper, we analyze the opposite situation, i.e., cases where the cycle class map is bijective. A large class of such varieties is given by "cellular" varieties (we leave the precise definition of "cellular" vague, but one may use, e.g., the stably cellular varieties of [17, Definition 2.10]). To focus the discussion, we formulate the following problem.

QUESTION 1. – *Let X be a smooth complex variety that is "cellular" (e.g., $\mathbb{P}^n$). Is every complex topological vector bundle motivic? In other words, for such an X is the map*

$$[X, \mathrm{Gr}_n]_{\mathbb{A}^1} \longrightarrow \mathscr{V}_n^{\mathrm{top}}(X)$$

surjective for every integer $n \geq 1$?

Our interest in this question has three sources. First, if X is furthermore affine, then Grauert's Oka principle establishes that every topological vector bundle on X admits a unique holomorphic structure, so the right hand side can be replaced by holomorphic vector bundles. In that setting, the question above is a special case of a question of Serre [34, §4 (3)].

Second, one may always analyze the problem of deciding whether a given rank r complex topological vector bundle on a smooth complex algebraic variety is algebraizable in two steps: decide if the given bundle is motivic, and, if it is, decide whether it lies in the image of the map $\mathscr{V}_r(X) \to [X, \mathrm{Gr}_r]_{\mathbb{A}^1}$. The latter question may be phrased more concretely using a Jouanolou device of X. Indeed, a motivic vector bundle on X corresponds to an algebraic vector bundle on a Jouanolou device $(\tilde{X}, \pi)$ for X, and asking whether a bundle lies in the image of the map $\mathscr{V}_r(X) \to [X, \mathrm{Gr}_r]_{\mathbb{A}^1}$ is equivalent to asking whether the given bundle descends along π.

Third, it is a difficult problem to construct indecomposable algebraic vector bundles of rank r on $\mathbb{P}^n$ when $1 < r < n$; ranks in this range are called "small." The problem of constructing vector bundles of small rank was explicitly stated by Schwarzenberger in the 1960s and Mumford called the special case of rank 2 bundles on $\mathbb{P}^n$, $n \geq 5$, "the most interesting unsolved problem in projective geometry that [he knew] of" [28, p. 227]. In a slightly broader context, Evans and Griffith write [19, p. 113] that small rank bundles "seem to be rare in nature". In view of the preceding paragraph, one can view the task of constructing motivic vector bundles of small rank as explicitly producing algebraic vector bundles of small rank if X is affine, and as a preliminary step toward building small rank vector bundles on projective varieties.

REMARK 2. – The problem of algebraizability of vector bundles on $\mathbb{P}^n$ has been studied for $n \leq 3$ by various authors. The case $n = 1$ being immediate, Schwarzenberger resolved the case $n = 2$. In this case, topological vector bundles are classified by their Chern classes, which may be identified with pairs of integers, and Schwarzenberger [31, 32] constructed algebraic vector bundles with prescribed Chern classes. The algebraizability of vector bundles on $\mathbb{P}^3$ was more subtle. Schwarzenberger showed that rank 2 topological bundles on $\mathbb{P}^3$ had additional congruence conditions on Chern classes (stemming from the Riemann-Roch theorem). Horrocks [22] showed that there exist rank 2 vector bundles on $\mathbb{P}^3$ with given Chern classes satisfying this condition. Atiyah and Rees completed the topological classification of rank 2 vector bundles on $\mathbb{P}^3$ showing that there is an additional mod 2 invariant (the α-invariant) for rank 2 vector bundles with even first Chern class. They then showed that Horrocks' bundles actually provide algebraic representatives for each isomorphism class of rank 2 topological vector bundles. Various other topological classification results exist and we refer the reader to [29] for more details.

REMARK 3. – Let k be a field (no longer assumed to be a subfield of $\mathbb{C}$). At the moment, we do not know a single example of a smooth k-variety X such that the map $\mathscr{V}_n(X) \to [X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ is not surjective. If X is a smooth projective curve over k, then it is straightforward to show that $\mathscr{V}_n(X) \to [X, \mathrm{Gr}_n]_{\mathbb{A}^1}$ is surjective for any integer n. We will show in future work that surjectivity can be guaranteed for smooth projective surfaces over an infinite field or smooth projective 3-folds over an algebraically closed field. For a general smooth k-variety X, note that if $(\tilde{X}, \pi)$ is a Jouanolou device for X, then $\tilde{X} \times_X \tilde{X}$ is again a smooth affine variety, and either projection $\tilde{X} \times_X \tilde{X} \to \tilde{X}$ is a torsor under a vector bundle, and therefore an isomorphism in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category. It follows that the two maps $\mathscr{V}_r(\tilde{X}) \to \mathscr{V}_r(\tilde{X} \times_X \tilde{X})$ coincide and are always bijections. Therefore, given any vector bundle $\mathscr{E}$ on $\tilde{X}$, there exists an isomorphism $\theta : p_1^*\mathscr{E} \xrightarrow{\sim} p_2^*\mathscr{E}$. The descent question is tantamount to asking whether θ may be chosen to satisfy the cocycle condition.

The precise goal of this paper is to analyze the algebraizability question (more precisely, Question 1) for a class of "interesting" topological vector bundles on $\mathbb{P}^n$ introduced by E. Rees and L. Smith. Let us recall the construction of these topological vector bundles following [30]; we refer to them as *Rees bundles* in the sequel. By a classical result of Serre [33, Proposition 11], we know that if p is a prime, then the p-primary component of $\pi_{4p-3}(S^3)$ is isomorphic to $\mathbb{Z}/p$, generated by the composite of a generator α_1 of the p-primary component of $\pi_{2p}(S^3)$ and the $(2p-3)$rd suspension of itself; we write α_1^2 for this class. In fact, $\pi_{4p-3}(S^3)$ is the first odd degree homotopy group of S^3 with non-trivial p-primary torsion. The map $\mathbb{P}^n \to S^{2n}$ that collapses $\mathbb{P}^{n-1}$ to a point determines a function

$$[S^{2n-1}, S^3] \cong [S^{2n}, BSU(2)] \longrightarrow [\mathbb{P}^n, BSU(2)]$$

Using the fact that $\pi_{4p-3}(S^3)$ is the first non-trivial p-torsion in an odd degree homotopy group of S^3, Rees showed that the class α_1^2 determines a non-trivial rank 2 vector bundle