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A MOTIVIC CIRCLE METHOD

BY Margaret BILU AND Tim BROWNING

ABSTRACT. – The circle method has been successfully used over the last century to study rational points on hypersurfaces. More recently, a version of the method over function fields, combined with spreading out techniques, has led to a range of results about moduli spaces of rational curves on hypersurfaces. In this paper a version of the circle method is implemented in the setting of the Grothendieck ring of varieties. This allows us to approximate the classes of these moduli spaces directly, without relying on point counting, and leads to a deeper understanding of their geometry.

RÉSUMÉ. – La méthode du cercle a été utilisée avec succès au cours du siècle dernier pour l'étude des points rationnels sur les hypersurfaces. Plus récemment, une version fonctionnelle de cette méthode, combinée à des techniques d'étalement, a mené à une série de résultats sur les espaces de modules de courbes sur les hypersurfaces. Dans cet article on implémente une version de la méthode du cercle dans le cadre de l'anneau de Grothendieck des variétés. Cela permet d'approximer les classes de ces espaces de modules directement, sans recours au comptage de points, ce qui donne accès à une compréhension plus profonde de leur géométrie.

1. Introduction

Let k be a field and let $f \in k[x_1, \dots, x_n]$ be a non-singular homogeneous polynomial of degree $d \geq 2$, defining a hypersurface $X \subset \mathbf{A}^n$. For global fields, the density of k-points on X has been the object of intense study over the years. When $k = \mathbf{Q}$, the Hardy-Littlewood circle method can be used to cast light on the limit

$$\lim_{B \to \infty} B^{-(n-d)} \#\{\mathbf{x} \in \mathbf{Z}^n : |\mathbf{x}| \leq B,\ f(\mathbf{x}) = 0\},$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and $|\mathbf{x}| = \max_{1 \leq i \leq n} |x_i|$. Thus, it follows from work of Birch [7] that this limit exists as a product of local densities, provided that $n > 2^d(d-1)$. Lee [32] has worked out the analogous statement when $k = \mathbf{F}_q(t)$, for a finite field $\mathbf{F}_q$ of

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characteristic $> d$. Thus, under the same assumption $n > 2^d(d-1)$, a similar statement is proved about the existence and nature of the limit

$$\lim_{e\to\infty} q^{-e(n-d)}\#\{\mathbf{g}\in \mathbf{F}_q[t]^n : \deg(g_1),\dots,\deg(g_n)\le e,\ f(\mathbf{g})=0\},$$

where $\mathbf{g}=(g_1,\dots,g_n)$. This paper is concerned with the field $k=\mathbf{C}(t)$ and the geometry of the affine variety

$$M_e=\{\mathbf{g}\in \mathbf{C}[t]^n : \deg(g_1),\dots,\deg(g_n)\le e,\ f(\mathbf{g})=0\}, \tag{1.1}$$

over $\mathbf{C}$, as $e\to\infty$. We have $(e+1)n$ indeterminates and $de+1$ equations, and so the expected dimension of M_e is

$$\mu(e):=(e+1)n-(de+1)=e(n-d)+n-1. \tag{1.2}$$

Typically, tools from algebraic geometry only allow one to prove that the expected dimension is actually the true dimension when X is *generic*. This follows from seminal work of Harris, Roth and Starr [24] when $d<n/2$, a restriction that has since been weakened to $d<2n/3$ by Beheshti and Kumar [3] (assuming that $n\ge 23$), and then to $d\le n-3$ by Riedl and Yang [41]. In the setting $d=3$ of cubic hypersurfaces it is possible to obtain results for *all* smooth hypersurfaces in the family. Thus Coskun and Starr [16] have shown that M_e is irreducible and of dimension $\mu(e)$, for any smooth cubic hypersurface $X\subset \mathbf{A}^n$, provided that $n\ge 4$. In the setting of smooth projective Fano varieties, the behavior of the invariants associated to spaces like M_e is explored from the perspective of the *geometric Manin conjecture* by Lehmann and Tanimoto [33].

It turns out that the circle method can shed light on the geometry of M_e when d is small enough compared to n, for any hypersurface $X\subset\mathbf{A}^n$ defined by a non-singular form. Thus, under the assumption $n>(5d-4)2^{d-1}$, it follows from [12] that M_e is irreducible and of dimension $\mu(e)$. This was later refined in [11, Thm. 1.2], with the outcome that for $n > 3\cdot 2^{d-1}(d-1)$, the variety M_e is a reduced and irreducible complete intersection of dimension $\mu(e)$, which is smooth outside a set of codimension at least $(n/2^{d-2}-6(d-1))(1+\lfloor\frac{e+1}{d-1}\rfloor)$. The arguments in BV,BSfree rely on *spreading out*, which translates the problem into a counting problem over a finite field $\mathbf{F}_q$. An application of the Lang-Weil estimates yields the desired conclusion via an application of the circle method over $\mathbf{F}_q(t)$.

This paper aims to tackle deeper questions about the geometry of M_e without recourse to spreading out and a reduction to counting arguments. A step in this direction was taken recently in [10], where a sheaf-theoretic version of the circle method is developed to study a variant of M_e, using tools from étale cohomology. Our goal is to rework the circle method in a motivic setting and to try and understand the class of M_e in the Grothendieck ring of varieties.

The *Grothendieck ring of varieties* $K_0(\mathrm{Var}_{\mathbf{C}})$ over $\mathbf{C}$ is defined by generators and relations. The generators are $\mathbf{C}$-varieties X. The relations are given by $X-Y$, whenever X and Y are $\mathbf{C}$-varieties that are isomorphic over $\mathbf{C}$, and the scissor relation $X-Y-U$, whenever X is a $\mathbf{C}$-variety, Y a closed subscheme of X and $U=X\setminus Y$ its open complement. We write $[X]$ for the class in $K_0(\mathrm{Var}_{\mathbf{C}})$ of X. The product $[X][Y]=[X\times_{\mathbf{C}} Y]$ endows $K_0(\mathrm{Var}_{\mathbf{C}})$ with a ring structure. We denote by $\mathbf{L}$ the class of $\mathbf{A}^1$ in $K_0(\mathrm{Var}_{\mathbf{C}})$, which is a zero divisor in the ring, by work of Borisov [8]. Inverting $\mathbf{L}$ gives the ring $\mathscr{M}_{\mathbf{C}}=K_0(\mathrm{Var}_{\mathbf{C}})[\mathbf{L}^{-1}]$. We are interested in the class of M_e in $\mathscr{M}_{\mathbf{C}}$. In fact, since the circle method involves working with exponential sums, in order to be able to set up a motivic version of it, we need to work in the *Grothendieck ring of varieties with exponentials* $K_0(\mathrm{ExpVar}_{\mathbf{C}})$ and its localization $\mathscr{E}xp\mathscr{M}_{\mathbf{C}} = K_0(\mathrm{ExpVar}_{\mathbf{C}})[\mathbf{L}^{-1}]$. The definition of $K_0(\mathrm{ExpVar}_{\mathbf{C}})$ is given in Section 2, together with some of its properties. In order to talk about convergence, as $e \to \infty$, we need to introduce a topology on $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$, which refines the obvious filtration by dimension. In Section 4 we carry out the construction of an appropriate weight function

$$w : \mathscr{E}xp\mathscr{M}_{\mathbf{C}} \to \mathbf{Z},$$

coming from Hodge theory. We denote by $\widehat{\mathscr{M}_{\mathbf{C}}}$ the completion of $\mathscr{M}_{\mathbf{C}}$ with respect to this weight topology.

Working in $\mathscr{E}xp\mathscr{M}_{\mathbf{C}}$, our main result shows that the class of M_e converges to the classes of local terms that resemble the main term in the circle method, as $e \to \infty$. Given $x \in \mathbf{A}^1$ and integer $N \geq 0$, our work involves the jet spaces

$$\Lambda_N(f, x) = \left\{\mathbf{g} \in \mathbf{C}[t]^n : \deg(g_1), \dots, \deg(g_n) < N,\ f(\mathbf{g}) \in (t-x)^N \mathbf{C}[t]\right\}, \tag{1.3}$$

parameterising solutions of $f(\mathbf{g}) = 0$ modulo $(t-x)^N$, together with

$$\Lambda_N(f, \infty) = \left\{\mathbf{g} \in \mathbf{C}[t]^n : \begin{array}{l} \deg(g_1), \dots, \deg(g_n) < N \\ f(g_1(t^{-1}), \dots, g_n(t^{-1})) \in t^{-N}\mathbf{C}[t^{-1}] \end{array}\right\}. \tag{1.4}$$

Note that $\Lambda_N(f, \infty)$ can naturally be identified with $\Lambda_N(f, 0)$. Bearing this notation in mind, we prove the following result.

THEOREM 1.1. – *Let $f \in \mathbf{C}[x_1, \dots, x_n]$ be a non-singular homogeneous polynomial of degree $d \geq 3$, defining a hypersurface $X \subset \mathbf{A}^n$. Assume $n > 2^d(d-1)$ and let $e \geq 1$. Then*

$$[M_e] = \mathbf{L}^{\mu(e)} \left(\mathfrak{S}(f) \cdot \lim_{N \to \infty} \mathbf{L}^{-N(n-1)}[\Lambda_N(f, \infty)] + R_e\right)$$

in $\widehat{\mathscr{M}_{\mathbf{C}}}$, where

$$\mathfrak{S}(f) = \prod_{x \in \mathbf{A}^1} \lim_{N \to \infty} \mathbf{L}^{-N(n-1)}[\Lambda_N(f, x)]$$

is a motivic Euler product and R_e is an error term satisfying

$$w(R_e) \leq 4 - \frac{n - 2^d(d-1)}{2^{d-2}} \left(1 + \left\lfloor \frac{e+1}{2d-2} \right\rfloor\right).$$

The term $\mathfrak{S}(f)$ is the *motivic singular series* and emerges as an infinite convergent sum, whose precise definition is given in (6.12). In Sections 6.3.3 and 6.3.4 we express $\mathfrak{S}(f)$ as a motivic Euler product, using the construction of motivic Euler products found in [5, Section 3]. The relevant background facts about motivic Euler products are recalled in Section 2.4.

REMARK 1.2. – In this paper the focus is on hypersurfaces of degree $d \geq 3$. However, on combining Propositions 3.7 and 3.8 in (3.1), it is also possible to deduce a statement for $d = 2$, with the only difference being a weaker error term if $e \leq n/2 - 7$.