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AN OBSTRUCTION TO THE LOCAL LIFTING PROBLEM

BY Aristides KONTOGEORGIS AND Alexios TEREZAKIS

Abstract. – We are investigating the lifting problem for local actions involving semidirect products of a cyclic p-group with a cyclic group prime to p, where p represents the characteristic of the special fiber. We establish a criterion based on the Harbater-Katz-Gabber compactification of local actions, enabling us to determine whether a given local action can be lifted or not. Specifically, in the case of the dihedral group, we present an example of a local dihedral action that cannot be lifted. This instance provides a more potent obstruction than the KGB obstruction.

Résumé. – Nous étudions le problème de relèvement des actions locales des produits semi-directs d'un groupe cyclique p par un groupe cyclique d'ordre premier avec p, où p est la caractéristique de la fibre spéciale. Nous obtenons un critère basé sur la compactification des actions locales de Harbater-Katz-Gabber, qui nous permet de décider si une action locale peut être relevée ou non. En particulier, dans le cas du groupe diédral, nous donnons un exemple d'action locale diédrale qui ne peut pas être relevée, offrant ainsi une obstruction plus forte que l'obstruction KGB.

1. Introduction

Let G be a finite group, k an algebraically closed field of characteristic $p > 0$ and consider the homomorphism

$$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]]),$$

which will be called *a local G-action*. Let $W(k)$ denote the ring of Witt vectors of k. The local lifting problem addresses the question: does there exist an extension $\Lambda/W(k)$, that is Λ is an integrally closed domain contained in a field extension of $\mathrm{Frac}(W(k))$, and a representation

$$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]]),$$

such that if t is the reduction of T, then the action of G on $\Lambda[[T]]$ reduces to the action of G on $k[[t]]$? If the answer to the above question is affirmative, then we say that the G-action lifts to characteristic zero. A group G for which every local G-action on $k[[t]]$ lifts to characteristic zero is called *a local Oort group for k*.

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Following an examination of specific obstructions, such as the Bertin obstruction, the KGB obstruction, and the Hurwitz tree obstruction, it has been established that the potential candidates for local Oort groups in characteristic p are limited to the following:

1. Cyclic groups;
2. Dihedral groups D_{p^h} of order $2p^h$;
3. The alternating group A_4.

The Oort conjecture states that every cyclic group C_q of order $q = p^h$ is a local Oort group. This conjecture was recently proven by F. Pop [20] using the work of A. Obus and S. Wewers [18]. A. Obus proved that A_4 is a local Oort group in [15] and this was also known to F. Pop as well as I. Bouw and S. Wewers [4]. Dihedral groups D_p are known to be local Oort by the work of I. Bouw and S. Wewers for p odd [4] and by the work of G. Pagot [19]. Several cases of dihedral groups D_{p^h} for small p^h have been studied by A. Obus [16] and H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte [9], while the D_4 was studied by B. Weaver [26]. For further details on the lifting problem, refer to [6, 7, 8, 14].

Perhaps the most significant of the currently known obstructions is the KGB obstruction [7]. It was conjectured that if the p-Sylow subgroup of G is cyclic, then this is the sole obstruction for the local lifting problem, see [14, 16]. In particular, the KGB obstruction for the dihedral group D_q is known to vanish, and the so called “generalized Oort conjecture” asserts that the local action of D_q always lifts for q-odd.

In this article, we will provide a new obstruction for the lifting problem of a $C_q \rtimes C_m$-action and in particular for the group D_q. We would like to emphasize that in contrast to the KGB obstruction, which vanishes for the dihedral groups of order $2p^h$, our obstruction does not. Using this criterion we provide in Section 5.1 a counterexample to the generalized Oort conjecture, by proving the HKG-cover corresponding to D_{125}, with a selection of lower jumps $9, 189, 4689$ does not lift.

We use the Harbater-Katz-Gabber compactification (referred to as HKG) as a means to construct complete curves from local actions. This approach equips us with a diverse set of tools stemming from the theory of complete curves, thereby allowing us to convert the local action and its deformations into representations of linear groups that act on the differentials of the HKG-curve. The foundational tools for this purpose are detailed in our article [12], where we have compiled various insights into the interrelation between lifting local actions, lifting curves, and lifting linear representations.

To elaborate further, let us delve into the specifics. Consider a local action $\rho : G \to \operatorname{Aut} k[[t]]$, where the group G is $C_q \rtimes C_m$. The Harbater-Katz-Gabber compactification theorem asserts that there is a Galois cover $X \to \mathbb{P}^1$ ramified wildly and completely only at one point P of X with Galois group $G = \operatorname{Gal}(X/\mathbb{P}^1)$ and tamely on a different point P' with ramification group C_m, so that the action of G on the completed local ring $\mathcal{O}_{X,P}$ coincides with the original action of G on $k[[t]]$. Notably, it is established that the local action lifts if and only if the corresponding HKG-cover undergoes lifting as well.

In particular, we have proved that in order to lift a subgroup $G \subset \operatorname{Aut}(X)$, the representation $\rho : G \to \operatorname{GL} H^0(X, \Omega_X)$ should be lifted to characteristic zero and also the lifting should be compatible with the deformation of the curve. More precisely, in [12] we have proved the following relative version of Petri’s theorem.

Proposition 1. – *Let $f_1,\dots,f_r \in S := \operatorname{Sym} H^0(X,\Omega_X) = k[\omega_1,\dots,\omega_g]$ be quadratic polynomials which generate the canonical ideal I_X of a curve X defined over an algebraic closed field k. Any deformation $\mathcal{X}_A$ is given by quadratic polynomials*

$$\tilde{f}_1,\dots,\tilde{f}_r \in \operatorname{Sym} H^0(\mathcal{X}_A,\Omega_{\mathcal{X}_A/A}) = A[W_1,\dots,W_g],$$

which reduce to $f_1,\dots,f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.

Additionally, we have provided the following liftability criterion:

Theorem 2. – *Consider an epimorphism $R \to k \to 0$ of local Artin rings. Let X be a curve which is canonically embedded in $\mathbb{P}_k^{g-1}$ and the canonical ideal is generated by quadratic polynomials; and acted on by the group G. The curve $X \to \operatorname{Spec}(k)$ can be lifted to a family $\mathcal{X} \to \operatorname{Spec}(R) \in D_{\mathrm{GL}}(R)$ along with the G-action, if and only if the representation $\rho_k : G \to \mathrm{GL}_g(k) = \mathrm{GL}(H^0(X,\Omega_X))$ lifts to a representation $\rho_R : G \to \mathrm{GL}_g(R) = \mathrm{GL}(H^0(\mathcal{X},\Omega_{\mathcal{X}/R}))$ and moreover the lift of the canonical ideal is left invariant by the action of $\rho_R(G)$.*

In Section 3 we prove that the canonical ideal of the HKG-cover is generated by quadratic polynomials, therefore Theorem 2 can be applied. In order to decide whether a linear representation of $G = C_q \rtimes C_m$ can be lifted, we will use the following criterion for the lifting of the linear representation, based on the decomposition of a $k[G]$-module into indecomposable summands. We begin by describing the indecomposable $k[G]$-modules for the group $G = C_q \rtimes C_m$:

Proposition 3. – *Suppose that the group $G = C_q \rtimes C_m$ is represented in terms of generators σ,τ and relations as follows:*

$$G = \langle \sigma,\tau \mid \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^\alpha\rangle,$$

for some $\alpha \in \mathbb{N}, 1 \le \alpha \le p^h - 1, (\alpha,p) = 1$. Every indecomposable $k[G]$-module has dimension $1 \le \kappa \le q$ and is of the form $V_\alpha(\lambda,\kappa)$, where the underlying space of $V_\alpha(\lambda,\kappa)$ has the set of elements $\{(\tau-1)^\nu e, \nu = 0,\dots,\kappa-1\}$ as a basis for some $e \in V_\alpha(\lambda,\kappa)$, and the action of σ on e is given by $\sigma e = \zeta_m^\lambda e$, for a fixed primitive m-th root of unity.

Proof. – A proof can be found in [11, sec. 3]. Notice also that $(\tau-1)^\kappa e = 0$. □

Notice that in Section 5 we will give an alternative description of the indecomposable $k[G]$-modules, namely the $U_{\ell,\mu}$ notation, which is compatible with the results of [3].

Remark 4. – In the article [11] of the authors, the $V_\alpha(\lambda,\kappa)$ notation is used. In this article we will need the Galois module structure of the space of holomorphic differentials of a curve and we will employ the results of [3], where the $U_{\ell,\mu}$ notation is used. These modules will be defined in Section 5, notice that $V_\alpha(\lambda,\kappa) = U_{(\lambda+a_0(\kappa-1)) \bmod m,\kappa}$, where $\alpha = \zeta_m^{a_0}$, see Lemma 14.

Theorem 5. – *Consider a $k[G]$-module M which is decomposed as a direct sum*

$$M = V_\alpha(\varepsilon_1,\kappa_1) \oplus \cdots \oplus V_\alpha(\varepsilon_s,\kappa_s).$$

The module lifts to an $R[G]$-module if and only if the set $\{1,\dots,s\}$ can be written as a disjoint union of sets I_ν, $1 \le \nu \le t$ so that