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A DEGENERATE ARNOLD DIFFUSION MECHANISM IN THE RESTRICTED 3-BODY PROBLEM

BY Marcel GUARDIA, Jaime PARADELA AND Tere M. SEARA

ABSTRACT. – A major question in dynamical systems is to understand the mechanisms driving global instability in the 3-body problem (3BP), which models the motion of three bodies under Newtonian gravitational interaction. The 3BP is called restricted if one of the bodies has zero mass and the other two, the primaries, have strictly positive masses m_0, m_1. We consider the restricted planar elliptic 3-body problem (RPE3BP) where the primaries revolve in Keplerian ellipses. We prove that the RPE3BP exhibits topological instability: for any values of the masses m_0, m_1, except $m_0 = m_1$, we build orbits along which the angular momentum of the massless body experiences an arbitrarily large variation provided the eccentricity of the orbit of the primaries is positive but small enough.

In order to prove this result we show that a degenerate Arnold diffusion mechanism, which moreover involves exponentially small phenomena, takes place in the RPE3BP. Our work extends the one of Delshams, Kaloshin, de la Rosa, and Seara (2019) for the a priori unstable case $m_1/m_0 \ll 1$, to the case of arbitrary masses $m_0, m_1 > 0$, where the model displays features of the so-called *a priori stable* setting.

RÉSUMÉ. – Une question importante en systèmes dynamiques est celle de comprendre les mécanismes responsables de l'instabilité globale dans le problème des trois corps, qui modélise le mouvement de trois corps soumis à l'interaction gravitationnelle newtonienne. Le problème des trois corps est dit restreint si l'un des corps a une masse nulle et les deux autres, appelés primaires, ont des masses strictement positives m_0, m_1. Nous considérons ici le problème restreint plan elliptique des trois corps (PRPE3C), dans lequel les primaires décrivent des ellipses képlériennes. Nous démontrons que le PRPE3C présente de l'instabilité topologique: pour toutes les valeurs des masses m_0, m_1 (sauf $m_0 = m_1$), nous construisons des orbites le long desquelles le moment angulaire du corps de masse nulle subit une variation arbitrairement grande, pourvu que l'excentricité de l'orbite des primaires soit positive mais suffisamment petite.

Pour prouver ce résultat, nous montrons qu'un mécanisme dégénéré de diffusion d'Arnold, impliquant de surcroît des phénomènes exponentiellement petits, se produit dans le PRPE3C. Notre travail étend celui de Delshams, Kaloshin, de la Rosa et Seara (2019), qui traitait le cas *a priori* instable $m_1/m_0 \ll 1$, au cas de masses arbitraires $m_0, m_1 > 0$, où le modèle présente des caractéristiques du cadre dit *a priori* stable.

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1. Introduction

The N-body problem models the motion of N bodies under mutual gravitational interaction. While the system is integrable for $N = 2$, understanding its global dynamics for $N \geq 3$ is probably one of the oldest and most challenging questions in dynamical systems. A major achievement in this direction was the proof of the existence of a positive measure set of quasiperiodic motions in the N-body problem. This result was first established by Arnold in [1], who gave a master application of the KAM technique to the case of 3 coplanar bodies. The proof was later extended to case $N \geq 3$ in the work of Féjoz and Herman [28] (see also [60, 15]). On the other hand, Herman conjectured in his ICM address [43] that the set of non-wandering points for the flow of the N-body problem is nowhere dense on every energy level for $N \geq 3$. This is in accordance with the general belief that the N-body problem, although strongly degenerate, displays the main features of a "typical" Hamiltonian system. This conjecture would imply topological instability for the N-body problem in a very strong sense.

The existence of topological instability in Hamiltonian systems was first investigated by Arnold in [2], where he constructed an example of a nearly integrable Hamiltonian in which this kind of behavior occurs. To that end, Arnold proposed a mechanism giving rise to unstable motions based on the existence of a transition chain of invariant tori: a sequence of invariant irrational tori which are connected by transverse heteroclinic orbits. This mechanism is nowadays called the *Arnold mechanism*. Arnold verified that this mechanism takes place in a cleverly built model usually referred to as the Arnold model, and he conjectured that topological instability is indeed a common phenomenon in the complement of integrable Hamiltonian systems [1]. Despite the enormous amount of research (see for example [50, 13, 24, 7, 8, 12, 33, 44, 35] and the references therein), the Arnold diffusion phenomenon, and more generally the dynamics in the complement of the KAM tori set, is still poorly understood, and even more poorly for real-analytic or non-convex Hamiltonians.

In [2], Arnold conjectured that the mechanism of instability based on the existence of transition chains *"is applicable to the general case (for example, to the problem of 3 bodies)"*. However, results concerning the existence of Arnold diffusion in the 3-body problem or related models are rather scarce (see [10, 22, 17, 16] and also [20, 29] for numerical-based results).

The 3-body problem is called "restricted" if one of the bodies has zero mass and the other two, the primaries, have strictly positive masses m_0, m_1. In this limit problem, the motion of the primaries is just a 2-body problem and the dynamics of the massless body is governed by the gravitational interaction with the primaries. In this work, we consider the case in which the primaries revolve around each other in Keplerian ellipses of eccentricity $\zeta \in (0, 1)$ and the massless body moves on the same plane as the primaries. This model, usually known in the literature as the restricted planar elliptic 3-body problem (RPE3BP), is a $2 + 1/2$ degrees-of-freedom Hamiltonian system. For $\zeta = 0$, i.e., for the restricted planar circular 3-body problem, the rotational symmetry prevents the existence of topological instability in nearly integrable settings, see [38] and Remark 17 below.

The goal of this paper is to prove that a *degenerate Arnold diffusion mechanism* takes place in the RPE3BP: we show that for any value of the masses of the primaries ($m_0 \neq m_1$), there exist orbits of the RPE3BP along which the angular momentum of the massless body experiences any predetermined drift provided the eccentricity of the orbits of the primaries is positive but small enough. Note that the angular momentum is a conserved quantity in the 2-body problem, which can be seen as a limit problem of the restricted 3-body problem when $m_1/m_0 \to 0$.

To the best of our knowledge the first complete proof of existence of Arnold diffusion in celestial mechanics was obtained in [22], in which the authors showed the existence of topological instability in the RPE3BP. Nevertheless, this result was established under the strong hypothesis $m_1/m_0 \ll 1$ (see Section 1.2 for a more precise description of the setting). Under this condition, the problem falls in the so-called a priori unstable regime for the study of Arnold diffusion and can be analyzed by means of classical perturbation theory (see Section 1.3). Our result extends the work in [22] to the case of *arbitrary masses* $m_0, m_1 > 0$, a setting in which the problem displays many features of the so-called *a priori stable* case.

1.1. Main result

Fix a Cartesian reference system with origin at the center of mass of the primaries and choose units so that the total mass of the primaries is equal to 1. In these coordinates, the primaries, which we denote by q_0 and q_1, move along Keplerian ellipses of eccentricity $\zeta \in (0,1)$ whose time parametrization reads

$$q_0(t) = \mu\varrho(t)(\cos f(t), \sin f(t)) \qquad q_1(t) = -(1-\mu)\varrho(t)(\cos f(t), \sin f(t)),$$

where $m_0 = 1-\mu$ and $m_1 = \mu \in (0, 1/2]$ are the masses of q_0 and q_1, $\varrho : \mathbb{T} \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) is the distance between the primaries and $f : \mathbb{T} \to \mathbb{T}$ is the so-called *true anomaly* (see [64]). The RPE3BP describes the motion of a massless body $q \in \mathbb{R}^2$ in the gravitational field generated by the primaries and it is governed by the second order differential equations

$$\ddot{q} = (1-\mu)\frac{q - q_0(t)}{|q - q_0(t)|^3} + \mu\frac{q - q_1(t)}{|q - q_1(t)|^3}. \tag{1.1}$$

It is a classical fact that the RPE3BP admits a Hamiltonian structure. Introducing p and E the conjugate momenta to q and t, and the gravitational potential

$$U(q,t) = \frac{1-\mu}{|q - q_0(t)|} + \frac{\mu}{|q - q_1(t)|} \tag{1.2}$$

the RPE3BP is a three degrees-of-freedom Hamiltonian system with respect to

$$\mathcal{H}(q,p,t,E) = \frac{|p|^2}{2} - U(q,t) + E$$

and the canonical symplectic structure in the extended phase space $T^*(\mathbb{R}^2 \times \mathbb{T})$. The following is our main result.

THEOREM 1.1. – *Let $G(q,p) = |q \wedge p|$ be the angular momentum of the massless body, let μ be the mass ratio and let $\zeta \in (0,1)$ be the eccentricity of the ellipse described by the primaries orbit. Then, for any $\mu \in (0, 1/2)$, there exists $G_* > 0$ such that, for any $\zeta \in (0, G_*^{-3})$ and any values G_1, G_2 satisfying*

$$G_* \leq G_1 < G_2 \leq \zeta^{-1/3},$$