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HECKE ACTIONS ON LOOPS AND PERIODS OF ITERATED SHIMURA INTEGRALS

BY RICHARD HAIN
WITH AN APPENDIX BY PHAM HUU TIEP

Dedicated to the memory of Yuri Manin

ABSTRACT. – We show that the classical Hecke correspondences T_N act on the free abelian group generated by the conjugacy classes of the modular group $\mathrm{SL}_2(\mathbb{Z})$ and the conjugacy classes of its profinite completion. We show that this action induces a dual action on the ring of class functions of a certain relative unipotent completion of the modular group. This ring contains all iterated integrals of modular forms that are constant on conjugacy classes. It possesses a natural mixed Hodge structure and, after tensoring with $\mathbb{Q}_\ell$, a natural action of the absolute Galois group. Each Hecke correspondence preserves this mixed Hodge structure and commutes with the action of the absolute Galois group. Unlike in the classical case, where Hecke correspondences are acting on modular forms, the algebra generated by these generalized Hecke operators is not commutative.

In the appendix, Pham Tiep proves that, for all primes $p \geq 5$, every irreducible character of $\mathrm{SL}_2(\mathbb{Z}/p^n)/(\pm\,\mathrm{id})$ appears in its conjugation action on the group algebra of $\mathrm{SL}_2(\mathbb{Z}/p^n)$, a result needed in the body of the paper.

RÉSUMÉ. – Nous montrons que les correspondances de Hecke classiques T_N agissent sur le groupe abélien libre engendré par les classes de conjugaison du groupe modulaire $\mathrm{SL}_2(\mathbb{Z})$ et les classes de conjugaison de sa complétion profinie. Nous montrons que cette action induit une action duale sur l'anneau des fonctions de classe d'une certaine complétion unipotente relative du groupe modulaire. Cet anneau contient toutes les intégrales itérées de formes modulaires qui sont constantes sur les classes de conjugaison. Il possède une structure de Hodge mixte naturelle et, après tensorisation avec $\mathbb{Q}_\ell$, une action naturelle du groupe de Galois absolu. Chaque correspondance de Hecke préserve cette structure de Hodge mixte et commute avec l'action du groupe de Galois absolu. Contrairement au cas classique, où les correspondances de Hecke agissent sur les formes modulaires, l'algèbre engendrée par ces opérateurs de Hecke généralisés n'est pas commutative.

En annexe, Pham Tiep prouve que, pour tous les nombres premiers $p \geq 5$, tout caractère irréductible de $\mathrm{SL}_2(\mathbb{Z}/p^n)/(\pm\,\mathrm{id})$ apparaît dans son action par conjugaison sur l'algèbre de groupe de $\mathrm{SL}_2(\mathbb{Z}/p^n)$, un résultat utilisé dans le corps de l'article.

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1. Introduction

In this paper we show that the action of Hecke correspondences on invariants of modular curves, such as their cohomology groups, lifts to some non-abelian invariants. More precisely, we show that the classical Hecke correspondences T_N ($N \in \mathbb{N}_+$) act on the free abelian groups generated by the conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ and the conjugacy classes of its profinite completion. We show that this action induces a dual action on those iterated integrals of modular forms (*iterated Shimura integrals* in Manin's terminology [29]) that are constant on conjugacy classes, and also on their non-holomorphic generalizations. These form a ring of class functions on $\mathrm{SL}_2(\mathbb{Z})$ which possesses a natural mixed Hodge structure. Each such Hecke operator preserves this mixed Hodge structure and commutes with the action of the absolute Galois group on the ℓ-adic analogue of this ring. Unlike in the classical case, the algebra generated by these generalized Hecke operators is not commutative.

The problem of defining a Hecke action on iterated Shimura integrals was posed by Manin in [29, §3.3] where he writes:

> *The problem of extending these results to the iterated case remains a major challenge. One obstacle is that correspondences (in particular, Hecke correspondences) do not act directly on the fundamental groupoid (as opposed to the cohomology) and hence do not act on the iterated integrals which provide homomorphisms of this groupoid.*

An initial attempt to define a Hecke action on iterated Shimura integrals was made by him in [30, §5.2]. Restricting our attention to conjugation invariant iterated integrals circumvents the problem of base points.

The overall goal of the project is to use this Hecke action to understand periods of iterated Shimura integrals and extensions in the categories of mixed Hodge structures and ℓ-adic Galois representations of the form

$$0 \to \big(\mathrm{Sym}^{r_1} V_{f_1} \otimes \cdots \otimes \mathrm{Sym}^{r_m} V_{f_m}\big)(d) \to E \to \mathbb{Q}(0) \to 0 \tag{1}$$

that occur in subquotients of the coordinate ring of relative unipotent completions of modular groups. Here $f_1, \dots, f_m$ are Hecke eigen cusp forms, V_{f_j} the simple $\mathbb{Q}$-Hodge structure or ℓ-adic Galois representation that corresponds to f_j, and $\mathrm{Sym}^{r_j} V_{f_j}$ its r_j-th symmetric power. The extensions (1) are expected to be the Hodge and ℓ-adic realizations of Voevodsky motives.

The coordinate rings of such relative completions contain all iterated Shimura integrals. It is known by the work of Francis Brown on *multiple modular values* [4, Ex. 17.6] that all of the Hodge extensions of $\mathbb{Q}$ by $V_f(d)$ predicted by the conjectures of Beilinson [2, Conj. 3.4a] do occur in the coordinate ring of the standard relative completion of $\mathrm{SL}_2(\mathbb{Z})$. This is implied by the fact that the periods of twice iterated integrals of Eisenstein series can contain non-critical L-values of cusp forms, which was proved by Brown in [4]. It is hoped that all of the extensions (1) predicted by Beilinson's conjectures, and not excluded by Brown's observation [4, §17], occur in these coordinate rings.

The construction of the Hecke action on conjugacy classes is elementary and natural. Denote the set of conjugacy classes of a discrete (or profinite) group Γ by $\boldsymbol{\lambda}(\Gamma)$ and by $\mathbb{k}\boldsymbol{\lambda}(\Gamma)$ the free $\mathbb{k}$-module generated by it, where $\mathbb{k}$ is a commutative ring. The central objects of this paper are $\Bbbk\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$ and, dually, the functions $\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}$ that arise from conjugation-invariant iterated integrals of modular forms.

Before proceeding, it is worth recalling the relation between conjugacy classes in $\mathrm{SL}_2(\mathbb{Z})$ and closed geodesics on the modular curve, which we regard as an orbifold, or more accurately, a stack. The map

$$\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z})) \to \boldsymbol{\lambda}(\mathrm{PSL}_2(\mathbb{Z}))$$

is 2-to-1 with fibers $\{\gamma, -\gamma\}$. Apart from the two conjugacy classes of elements of order 4 of $\mathrm{SL}_2(\mathbb{Z})$, the two preimages of $\mu \in \boldsymbol{\lambda}(\mathrm{PSL}_2(\mathbb{Z}))$ are distinguished by the signs of their traces. The conjugacy classes of non-torsion elements of $\mathrm{PSL}_2(\mathbb{Z})$ correspond to powers of oriented closed geodesics in the modular curve and powers of the horocycle. So a non-torsion conjugacy class of $\mathrm{SL}_2(\mathbb{Z})$ corresponds to either a (not necessarily prime) closed geodesic or a non-zero power of the horocycle on the modular curve, together with the sign of its trace. This is explained in more detail in Section 8.

Theorem 1. – *The classical Hecke correspondences T_N, $N \in \mathbb{N}$, act on $\mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$. The operators T_N and T_M commute when M and N are relatively prime. These actions of the T_N descend to $\mathbb{Z}\boldsymbol{\lambda}(\mathrm{PSL}_2(\mathbb{Z}))$.*

This is proved in Section 9. The basic observation behind the existence of this Hecke action is that if $\pi : Y \to X$ is a finite unramified cover of topological spaces, then there are pushforward and pullback maps

$$\pi_* : \mathbb{Z}\lambda(Y) \to \mathbb{Z}\lambda(X) \text{ and } \pi^* : \mathbb{Z}\lambda(X) \to \mathbb{Z}\lambda(Y),$$

where, for a topological space X, $\lambda(X)$ denotes the set of free homotopy classes of maps from the circle to X. When X is path connected, $\lambda(X) = \boldsymbol{\lambda}(\pi_1(X, x))$. The pushforward map is simply composition with π; the pullback map takes a loop in X to the sum of closed loops in Y that cover its preimage under π. The precise definition can be found in Section 4.1. (This notion also occurs independently in [38], where it is called a *transfer map*.) In particular, this gives a definition of pushforward and pullback maps

$$\pi_* : \mathbb{Z}\boldsymbol{\lambda}(\Gamma') \to \mathbb{Z}\boldsymbol{\lambda}(\Gamma) \text{ and } \pi^* : \mathbb{Z}\boldsymbol{\lambda}(\Gamma) \to \mathbb{Z}\boldsymbol{\lambda}(\Gamma')$$

associated with the inclusion $\pi : \Gamma' \hookrightarrow \Gamma$ of a finite index subgroup by taking X and Y to be appropriate models of their classifying spaces $B\Gamma$ and $B\Gamma'$. One finds that

$$\pi_*\pi^*(\gamma) = \sum_j \gamma^{m_j} \qquad \gamma \in \boldsymbol{\lambda}(\Gamma),$$

where the m_j are positive integers that depend on γ and whose sum is the degree of π, which is the index of Γ' in Γ.

When p is a prime number, the (generalized) Hecke operator

$$T_p : \mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$$

is the map

$$\mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z})) \xrightarrow{\pi^*} \mathbb{Z}\boldsymbol{\lambda}(\Gamma_0(p)) \xrightarrow{\pi_*^{\mathrm{op}}} \mathbb{Z}\boldsymbol{\lambda}(\mathrm{SL}_2(\mathbb{Z}))$$