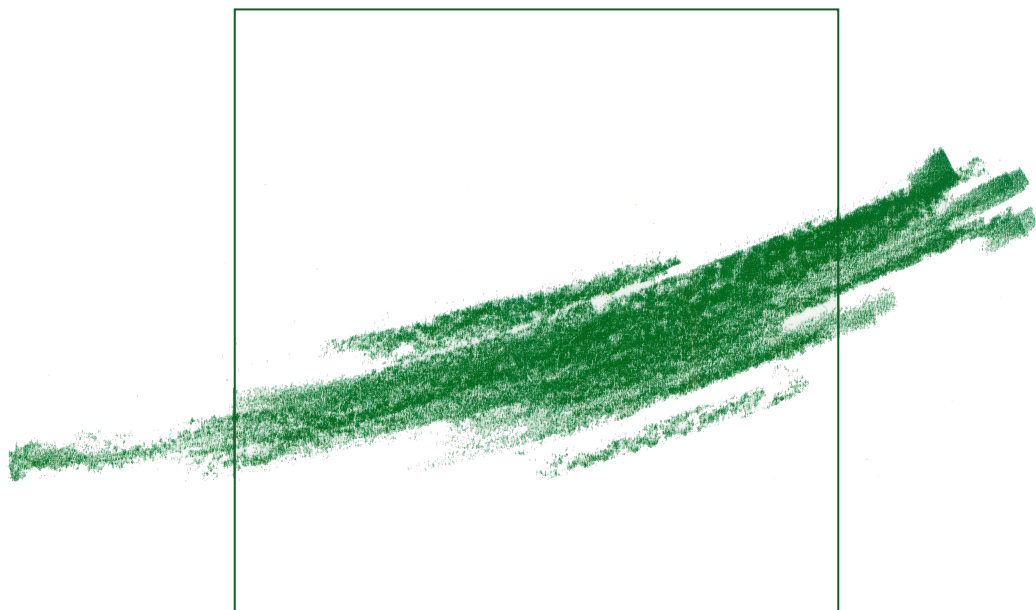


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Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23

INTEGRAL GEOMETRY FROM BUFFON TO GEOMETERS OF TODAY

Rémi Langevin

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Rémi Langevin

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