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The boundary values of generalized Dirichlet series and a problem of Chebyshev

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1. Introduction and statement of results

In 1853 Chebyshev asserted in a letter to M. Fuss that there are more primes $p \equiv 3 \pmod 4$ than $p \equiv 1 \pmod 4$. S. Knapowski and P. Turán in their well-known series of papers on comparative prime number theory [5] write, after quoting Littlewood's result that $\pi(x,4,1)-\pi(x,4,3)$ changes sign infinitely many times as $x \to \infty$, the following lines : *one feels that Chebyshev's vague formulation could also be interpreted so as*

$$(1.1) \qquad \lim_{Y\to\infty} N(Y)/Y = 0,$$

where $N(Y)$ denotes the number of integers $m \le Y$ with the property

$$(1.2) \qquad \pi(m,4,1) \ge \pi(m,4,3)$$

(cf. also [6], page 26). They support this conjecture by referring to Shanks [7], who found that (1.2) is not fulfilled for $m \le 26860$, is then fulfilled for $m = 26861$ and $m = 26862$, and is again false for $26863 \le m \le 616768$. They also ask the following general question ([5], Problem 7).

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For fixed positive integers a, b and q such that $(a,q) = (b,q) = 1$, $a \not\equiv b \pmod q$, what is the asymptotical behaviour of $N_{a,b}(Y)$ for $Y \to \infty$, where $N_{a,b}(Y)$ denotes the number of integers $m \le Y$ with

$$\pi(m,q,a) \ge \pi(m,q,b) \quad ?$$

Our aim is to prove a general result concerning boundary values of Dirichlet series and to show its relevance to Chebyshev's problem. As a corollary we obtain the following theorem.

THEOREM 1. *Suppose a and q are positive integers satisfying $(a,q) = 1$, $a \not\equiv 1 \pmod q$ and let the Generalized Riemann Hypothesis (G.R.H.) be true for Dirichlet L-series $\pmod q$. Then there exist two constants $0 < c_1 < c_2 < 1$ such that the inequalities*

$$c_1 Y \le N_{a,1}(Y) \le c_2 Y$$

hold for all sufficiently large Y.

This shows that the Knapowski-Turán conjecture (1.1) is false at least when we accept the G.R.H.

The basic tool used in the proof of Theorem 1 is a result concerning generalized Dirichlet series which seems to be of an independent interest. For the sake of brevity, let $\mathcal{A}$ denote the set of all functions

$$F(z) = \sum_{n=1}^{\infty} a_n e^{iw_n z}, \quad z = x + iy, \quad y > 0 \tag{1.3}$$

satisfying the following conditions:

1. $0 \le w_1 < w_2 < \dots$ are real numbers.
2. $a_n \in \mathbb{C}$, $n = 1, 2, 3, \dots$
3. There exists a non-negative integer B such that

$$\sum_{n=2}^{\infty} |a_n| w_n^{-B} < \infty. \tag{1.4}$$

4. There exists a non-negative number L_0 such that for every x, $|x| \ge L_0$, the limit
$$P(x) = \lim_{y \to 0+} \operatorname{Re} F(x+iy)$$
exists and represents a locally bounded function of $x \in \mathbb{R} \setminus [-L_0, L_0]$.

Moreover, let

$$\alpha(F) = \inf_{\substack{y>0 \\ x\in\mathbb{R}}} \operatorname{Re} F(x+iy), \qquad \beta(F) = \sup_{\substack{y>0 \\ x\in\mathbb{R}}} \operatorname{Re} F(x+iy).$$

It was proved in [4] that if $F \in \mathcal{A}$ and $\alpha(F) < u < \beta(F)$ then there exists a positive number $l = l(u, F)$ such that

$$\inf_{x\in I} P(x) < u < \sup_{x\in I} P(x) \tag{1.5}$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$.

This result is of importance to the prime number theory being a substitute for Ingham's method [1], [2]. Now we impose somewhat stronger conditions on F and we estimate the measure of the set of x satisfying (1.5).

THEOREM 2. *Let $F \in \mathcal{A}$ and suppose that*

$$\|P\|^2 = \sup_{|t|>L_0+1} \int_0^1 |P(x+t)|^2\, dx < \infty. \tag{1.6}$$

Then for every real number u satisfying $\alpha(F) < u < \beta(F)$ there exist positive constants $l = l(u,F)$ and $d_1 = d_1(u,F)$ such that

$$|\{x \in I \ :\ P(x) > u\}| \geq d_1 \tag{1.7}$$

and

$$|\{x \in I \ :\ P(x) < u\}| \geq d_1 \tag{1.8}$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$ (where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$).

We apply this theorem to the function

$$\begin{aligned} F_{a,b}(z) = -2e^{-z/2}\frac{1}{\phi(q)} &\sum_{\chi \ (\mathrm{mod}\ q)} (\overline{\chi(a)} - \overline{\chi(b)})\, K(z,\chi') \\ -\frac{2}{\phi(q)} &\sum_{\chi \ (\mathrm{mod}\ q)} (\overline{\chi(a)} - \overline{\chi(b)})\, m(\tfrac{1}{2},\chi), \end{aligned} \tag{1.9}$$

where $q \geq 2$, $0 < a, b < q$, $(a,q) = (b,q) = 1$, $a \not\equiv b \pmod q$ are integers, K denotes the K-function as introduced in [3]:

$$K(z,\chi') = \sum_{\gamma>0} e^{\rho z}, \quad z = x + iy, \quad y > 0$$

(the summation being taken over all non-trivial $L(s,\chi')$ zeros ρ with positive imaginary parts γ); χ' is the primitive Dirichlet character induced by χ, and $m(\frac{1}{2},\chi)$ is the multiplicity of a zero of $L(s,\chi)$ at $s=\frac{1}{2}$ (we put $m(\frac{1}{2},\chi)=0$ when $L(s,\chi)\neq 0$). We obtain the following corollaries.

COROLLARY 1. *Suppose the G.R.H. is true for Dirichlet L-functions* (mod q). *Then for every real number u satisfying $\alpha(F_{a,b})<u<\beta(F_{a,b})$ there exist positive constants $c_0=c_0(u,q)$ and $d_0=d_0(u,q)$ such that*

$$\left|\left\{T\le t\le c_0T\ :\ \psi(t,q,a)-\psi(t,q,b)>u\sqrt{t}\right\}\right|\ge d_0T \tag{1.10}$$

and

$$\left|\left\{T\le t\le c_0T\ :\ \psi(t,q,a)-\psi(t,q,b)<u\sqrt{t}\right\}\right|\ge d_0T \tag{1.11}$$

for sufficiently large T.

COROLLARY 2. *Suppose the G.R.H. is true for Dirichlet L-functions* (mod q) *and let $(a,q)=1$, $a\not\equiv 1$* (mod q). *Then for every positive u there exist $c_1=c_1(u,q)>0$ and $d_1=d_1(u,q)>0$ such that*

$$\#\left\{Y\le m\le c_1Y\ :\ \psi(m,q,a)-\psi(m,q,1)>u\sqrt{m}\right\}\ge d_1Y, \tag{1.12}$$

$$\#\left\{Y\le m\le c_1Y\ :\ \psi(m,q,a)-\psi(m,q,1)<-u\sqrt{m}\right\}\ge d_1Y, \tag{1.13}$$

$$\#\left\{Y\le m\le c_1Y\ :\ \pi(m,q,a)-\pi(m,q,1)>u\sqrt{m}/(\log m)\right\}\ge d_1Y, \tag{1.14}$$

and

$$\#\left\{Y\le m\le c_1Y\ :\ \pi(m,q,a)-\pi(m,q,1)<-u\sqrt{m}/(\log m)\right\}\ge d_1Y, \tag{1.15}$$

for all sufficiently large Y.

Let us remark that our Theorem 1 follows at once from Corollary 2; it is sufficient therefore to prove this corollary only.

Applying Theorem 2 to the function

$$F_a(z)=-2e^{-z/2}\frac{1}{\phi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi(a)}\,K(z,\chi')-\frac{2}{\phi(q)}\sum_{\chi\ (\mathrm{mod}\ q)}\overline{\chi(a)}\,m(\tfrac{1}{2},\chi),$$
$$(z=x+iy,\quad y>0,\quad (a,q)=1)$$