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# HOMOCLINIC ORBITS NEAR SADDLE-CENTER FIXED POINTS OF HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

*by*

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*To Jacob Palis for his* $60^{th}$ *Birthday*

***Abstract.*** — We study a class of Hamiltonian systems on a 4 dimensional symplectic manifold which have a saddle-center fixed point and satisfy the following property: All the periodic orbits in the center manifold of the fixed point have an orbit homoclinic to them, although the fixed point itself does not. In addition, we prove that these systems have a chaotic behavior in the neighborhood of the energy shell of the fixed point.

## Introduction

A fixed point of a Hamiltonian system with two degrees of freedom is called a Saddle-Center if the linearized vector field has one pair of purely imaginary eigenvalues and one pair of non zero real eigenvalues. A saddle-center fixed point is surrounded by a two-dimensional invariant manifold, the center manifold, filled by closed orbits. A saddle-center fixed point has also a one-dimensional stable manifold and a one-dimensional unstable manifold; the periodic orbits in the center manifold have two-dimensional stable and unstable manifolds. If a point belongs to the intersection of the stable and unstable manifold of the fixed point (resp. of one periodic orbit) then its orbit is biasymptotic to the fixed point (resp. the periodic orbit). We call such an orbit homoclinic.

Some consequences of the existence of an orbit homoclinic to the fixed point have been investigated in [**5**], [**9**], [**7**], [**8**], [**11**], [**18**] (specially section 7.2) and other papers. It should be noted, however, that the existence of such a homoclinic is exceptional, in contrast to the case of hyperbolic fixed points. Dimensional considerations show that orbits homoclinic to the periodic motions of the center manifold are more likely

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to exist. The existence of such homoclinics has been studied in [**4**], [**14**] (see also [**11**], [**9**], [**10**], [**7**], [**12**]) by perturbation methods, and in [**2**] by global methods. In these papers, orbits homoclinic to periodic orbits sufficiently far away from the fixed point are found.

In the present work, we study analytic perturbations of an integrable system with a homoclinic loop. We prove the following interesting behavior : Given any periodic orbit sufficiently close to the equilibrium in the center manifold, there exists an orbit homoclinic to it, although in general there does not exist any orbit homoclinic to the fixed point. This illustrates a question asked in [**2**].

In addition, topological entropy near the energy shell of the fixed point is obtained as a consequence of the presence of these homoclinics. More precisely, we prove that every neighborhood of the energy shell of the fixed point contains an energy shell with chaotic behavior on it. A similar result for reversible Hamiltonian systems is claimed, with no proof, in [**14**] pg 116. Other results in this direction under the hypothesis of the system being far from integrable can be found in [**9**], [**7**], [**13**].

Our method is semi-global and heavily relies on the low dimension: We first use the perturbative setting to prove the existence of quasiperiodic invariant tori confining the system in a neighborhood of the unperturbed homoclinic loop. We then reduce the problem to an area preservation argument on appropriate Poincaré return maps. It would of course be very interesting to obtain similar results by global methods and in higher dimension, in the spirit of [**2**], and to understand to what extent the phenomenon described here is general.

This paper emanated from a discussion between the authors after a talk of one of them at the international conference on dynamical systems dedicated to Jacob Palis. The authors would like to thank the organizers of that conference, who made that encounter possible. The first author learned a lot during his numerous conversations with Michel Herman, and was moved a lot by his sudden death.

## 1. Notations and results

**1.1.** Let $M$ be a four-dimensional analytic manifold, endowed with a symplectic form $\Omega$, and let

$$H : M \times I \longrightarrow \mathbb{R},$$
$$(x,\mu) \longmapsto H(x,\mu) = H_\mu(x)$$

be an analytic one-parameter family of Hamiltonians, where $I$ is some interval containing 0 in its interior. In all this paper, we shall assume that the Hamiltonian system $H_\mu$ has a saddle-center fixed point $r_\mu$ for all $\mu \in I$, and that $H_\mu(r_\mu) = 0$. It is by now classical (see [**15**], [**17**], [**5**], [**14**], [**7**]), that the system $H_\mu$ is integrable in the neighborhood of the saddle-center $r_\mu$. More precisely, there exist a neighborhood $U$ of 0 in $\mathbb{R}^4$ and an analytic mapping $\phi : U \times I \to M$ such that $\phi_\mu$ is a symplectic embedding for each $\mu$, $\phi_\mu(0) = r_\mu$, and

$$H_\mu \circ \phi_\mu(q_1, p_1, q_2, p_2) = h(I_1, I_2, \mu),$$

where

$$I_1 = p_1 q_1, \quad I_2 = (p_2^2 + q_2^2)/2,$$

and the function $h$ is analytic (one may have to reduce $I$). Furthermore, one can be reduced via a change in time-scale and a canonical transformation to the case where

$$\partial_{I_1} h(0,0,\mu) = -1 \quad \text{and} \quad \partial_{I_2} h(0,0,\mu) = \omega(\mu) > 0.$$

The functions $I_1$ and $I_2$ are preserved by the flow restricted to the local chart, this flow is determined by the equations

$$\begin{aligned} \dot p_1 &= -\partial_{I_1} h(I_1, I_2, \mu) p_1 & \dot p_2 &= -\partial_{I_2} h(I_1, I_2, \mu) q_2 \\ \dot q_1 &= \phantom{-}\partial_{I_1} h(I_1, I_2, \mu) q_1 & \dot q_2 &= \phantom{-}\partial_{I_2} h(I_1, I_2, \mu) p_2. \end{aligned}$$

It follows that the center manifold of $r_\mu$ has equation $I_1 = 0$, its stable manifold has equation $I_2 = 0$, $p_1 = 0$ and its unstable manifold $I_2 = 0$, $q_1 = 0$. In the following, we will call $P_{E,\mu}$ the periodic orbit of $H_\mu$ at energy $E$, which in local coordinates is the circle $p_1 = q_1 = 0$, $I_2 = E$.

**1.2.** We shall also suppose that $H_0$ is integrable (namely, its associated Hamiltonian vector field has an additional real analytic first integral $J$ such that $dH_0(x)$ and $dJ(x)$ are independent for almost every $x$) and that the vector field associated to $H_0$ has an orbit homoclinic to $r_0$ which connects the branch $p_1 > 0$ of the unstable manifold to the branch $q_1 > 0$ of the stable manifold. Integrable systems with a saddle-center and an orbit doubly asymptotic to it have been studied in [**9**], where it is explained that there exist two different kinds of homoclinics. For comparison, let us mention that we are here in case (A) of [**9**].

***1.3. Theorem***. — *Let us consider an analytic one-parameter family $H_\mu$ of Hamiltonian systems satisfying the above hypotheses. There exists a positive number $\varepsilon$ such that for all $E \in\, ]0, \varepsilon[$ and all $\mu \in\, ]-\varepsilon, \varepsilon[ \subset I$, there exists an orbit of $H_\mu$ homoclinic to the periodic orbit $P_{E,\mu}$. In fact, there even exist infinitely many geometrically distinct orbits homoclinic to $P_{E,\mu}$.*

***1.4. Theorem***. — *Let us fix $\mu \in\, ]-\varepsilon, \varepsilon[$. For each $E \in\, ]0, \varepsilon[$, either the stable and unstable manifolds of $P_{E,\mu}$ coincide, or the flow of $H_\mu$ on the energy shell $H_\mu = E$ has positive topological entropy.*

***1.5. Theorem***. — *Let us fix a value of $\mu$ satisfying the hypothesis of theorem 1.3. Assume in addition that the stable and unstable manifolds of the fixed point $r_\mu$ do not coincide. Then there exists a sequence $E_n > 0$ converging to $0$ and such that the stable and unstable manifolds of $P_{E_n,\mu}$ do not coincide. It follows that, for each $n$, the flow of $H_\mu$ restricted to the energy surface $H_\mu = E_n$ has positive topological entropy.*

**1.6.** The main result of the present paper is Theorem 1.3. It is proved in section 3. Theorem 1.4 may be considered classical. However we include a proof in section 4 because we could not find any reference matching precisely our needs. Theorem 1.5 is a simple but, we believe, interesting consequence. It is proved in section 5. The main notations and tools that will be used throughout the paper are introduced in section 2

***1.7. Remark***. — In order to apply Theorem 1.5, one has to be able to decide whether there exists an orbit homoclinic to the fixed point. Let us mention a result in that direction. Under an additional hypothesis of reversibility of the family of Hamiltonian systems $H_\mu$ (see [**7**]) it is possible to prove that the set of values of $\mu$ for which a homoclinic orbit to the equilibrium point $r_\mu$ occurs is either a whole interval or it is countable ([**7**], section 6). The same result may hold for the non reversible case considered here but this is an open question.

## 2. Local sections and invariant curves

We analyze the orbit structure near the homoclinic loop in a rather usual way (see [**5**], [**9**], [**14**],...), via Poincaré sections. More details in these papers. The existence of invariant curves was already obtained in [**8**].

**2.1.** Let us define the two Poincaré sections given in local coordinates by

$$\Sigma_1 = \{q_1 = \delta\}, \quad \Sigma_2 = \{p_1 = \delta\},$$

where $\delta$ is a small positive number. Since $\partial_{I_1} h = -1$, the equation $h(I_1, I_2, \mu) = E$ can be solved in $I_1$ for sufficiently small $I_2$, $E$ and $\mu$ *i.e.* there exists an analytic function $v$ defined in a neighborhood of 0 in $\mathbb{R}^3$ and such that

$$h(I_1, I_2, \mu) = E \iff I_1 = v(I_2, E, \mu).$$

As a consequence, for sufficiently small $E$ and $\mu$, the intersection $\Sigma_i(E,\mu)$ of $\Sigma_i$ with the energy shell $H_\mu = E$ is a graph over the $(p_2, q_2)$-plane. More precisely, the analytic mappings $\sigma_i^{E,\mu} : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\sigma_1^{E,\mu}(p_2, q_2) = \sigma_1(p_2, q_2, E, \mu) = \big(v\big(I_2(p_2, q_2), E, \mu\big)/\delta,\ \delta,\ p_2,\ q_2\big),$$
$$\sigma_2^{E,\mu}(p_2, q_2) = \sigma_2(p_2, q_2, E, \mu) = \big(\delta,\ v\big(I_2(p_2, q_2), E, \mu\big)/\delta,\ p_2,\ q_2\big)$$

are symplectic charts of $\Sigma_i(E,\mu)$. In the following, we note $y = (p_2, q_2)$ and take it as coordinates of $\Sigma_i(E,\mu)$.

**2.2.** The intersection between the stable manifold of $P_{E,\mu}$ and $\Sigma_1$, as well as the intersection between the unstable manifold and $\Sigma_2$, are the circles $I_2(y) = I^c(E,\mu)$ in coordinates, where $I^c(E,\mu)$ is the solution of the equation

$$h(0, I^c(E,\mu), \mu) = E \iff v(I^c(E,\mu), E, \mu) = 0.$$