The Shimura Subgroup of $J_0(N)$

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SUMMARY. — *To the natural morphism* $X_1(N) \to X_0(N)$ *of modular curves corresponds, by Picard functoriality, a morphism* $J_0(N) \to J_1(N)$ *between their Jacobian varieties. Its kernel* $\Sigma(N)$*, called the Shimura subgroup of* $J_0(N)$*, is finite. We determine the group structure of* $\Sigma(N)$ *together with the action of Galois and the action of the Hecke algebra. This extends previous results obtained by B. Mazur and K. Ribet.*

Let $N \geq 1$ be an integer and let $\Gamma_0(N)$ be the subgroup of $SL_2(\mathbf{Z})$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ such that N divides c. It acts on the Poincaré half-plane $\mathcal{H} = \{\tau \in \mathbf{C} |\ \operatorname{Im} \tau > 0\}$ and on $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

The quotient $X_0(N) = \Gamma_0(N)\backslash\overline{\mathcal{H}}$ has a natural structure of compact connected Riemann surface.

One defines in a similar way a Riemann surface $X_1(N) = \Gamma_1(N)\backslash\overline{\mathcal{H}}$, where $\Gamma_1(N)$ is the subgroup of $\Gamma_0(N)$ consisting of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a \equiv d \equiv 1 \bmod N$. Let $u : X_1(N) \to X_0(N)$ be the holomorphic map deduced from the identity on $\overline{\mathcal{H}}$ by passing to the quotients.

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Let $J_0(N)$ and $J_1(N)$ be the Jacobian varieties of $X_0(N)$ and $X_1(N)$, viewed as the connected components of 0 in the corresponding Picard varieties. Let

$$u^* : J_0(N) \longrightarrow J_1(N)$$

be the morphism of abelian varieties deduced from u by Picard functoriality. Its kernel, called the *Shimura subgroup* of $J_0(N)$, is a finite group; we denote it by $\Sigma(N)$.

In this paper, we give a complete description of $\Sigma(N)$: group structure, exponent, order, action of Galois, of Atkin-Lehner involutions and of Hecke operators (including those associated to the primes dividing N), behaviour under degeneracy maps, etc. This extends previous results obtained by B. Mazur ([3], II, 11) and K. Ribet ([5]). Our proofs are of complex analytic nature and would apply in situations where $\Gamma_0(N)$ and $\Gamma_1(N)$ are replaced by discrete subgroups of $SL_2(\mathbf{R})$ of finite covolume, even when the corresponding Riemann surfaces have no modular interpretation.

Let $\mathbf{U}$ be the group of complex numbers of modulus 1. We define in §1 a canonical injective group homomorphism

$$\psi : J_0(N) \longrightarrow \mathrm{Hom}(\Gamma_0(N), \mathbf{U}). \qquad (1)$$

Throughout the paper, we identify the group $\Gamma_0(N)/\Gamma_1(N)$ with $(\mathbf{Z}/N\mathbf{Z})^\times$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_1(N) \mapsto d + N\mathbf{Z}.$$

We show that an element x of $J_0(N)$ belongs to the Shimura subgroup $\Sigma(N)$ if and only if the kernel of $\psi(x)$ contains $\Gamma_1(N)$. Therefore, we deduce from ψ a canonical injective homomorphism

$$\psi' : \Sigma(N) \longrightarrow \mathrm{Hom}((\mathbf{Z}/N\mathbf{Z})^\times, \mathbf{U}). \qquad (2)$$

We determine its image in §2 and obtain:

THEOREM 1 .— *The Shimura subgroup $\Sigma(N)$ of $J_0(N)$ is canonically isomorphic to the group of homomorphisms $g : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{U}$ such that $g(d) = 1$ if $d = -1$, $d^2 + 1 = 0$, $d^2 + d + 1 = 0$ or $(d-1)^2 = 0$.*

By using thm. 1, we compute in §3 the order and the exponent of the group $\Sigma(N)$:

COROLLARY 1 .— *Let $\phi(N)$ denote the number of elements of* $(\mathbf{Z}/N\mathbf{Z})^\times$ *and:*

(i) let m be the largest integer such that m^2 divides N;

(ii) let k be the number of prime divisors of N distinct from 2 and 3;

(iii) let m_2 be equal to 2 if -1 is a square mod N *(i.e., if $4 \nmid N$ and each prime factor $p \neq 2$ of N is congruent to* 1 mod 4*), and let m_2 be equal to 1 otherwise;*

(iv) let m_3 be equal to 3 if X^2+X+1 has a root mod N *(i.e., if $9 \nmid N$ and each prime factor $p \neq 3$ of N is congruent to* 1 mod 3*), and let m_3 be equal to 1 otherwise.*

Then we have

$$\mathrm{Card}(\Sigma(N)) = \begin{cases} \phi(N)/(2mm_2^k m_3^k) & \text{if } N \geq 5 \\ 1 & \text{if } N \leq 4. \end{cases}$$

EXAMPLE.— If N is of the form p^n, with p a prime number and $n \geq 1$, then $\Sigma(N)$ is a cyclic group (thm. 1). If $p \neq 2$, its order is the product of $p^{n-1-[n/2]}$ and the numerator of $\frac{p-1}{12}$; if $p = 2$, its order is $2^{\max(0,n-2-[n/2])}$.

COROLLARY 2 .— *Let $N = \prod p^{r_p}$ be the prime power decomposition of N and:*

(i) let r'_p be equal to $r_p - 1 - [r_p/2]$ if $p \neq 2$;

(ii) let r'_2 be equal to $\max(0, r_2 - 2 - [r_2/2])$*;*

(iii) let e_0 be equal to $\operatorname*{lcm}_{p|N}((p-1)p^{r'_p})$*;*

(iv) let m_1 be equal to 2 if N is the product of $1, 2$ or 4 by a power of an odd prime, and let m_1 be equal to 1 otherwise;

(v) let m_2 and m_3 be as in cor. 1.

Then the exponent of the group $\Sigma(N)$ (i.e., the smallest integer e such that $e\Sigma(N) = 0$) is given by

$$e = \begin{cases} e_0/(m_1 m_2 m_3) & \text{if } N \geq 5 \\ 1 & \text{if } N \leq 4. \end{cases}$$

COROLLARY 3 .— *The only integers N for which the order of $\Sigma(N)$ is* 1 *are* 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25, 36, 49, 50 *and* 169.

In fact, for all these values of N except 36, 49, 50 and 169, the genus of the Riemann surface $X_0(N)$ is 0 and we therefore have $J_0(N) = 0$.

COROLLARY 4 .— *When N approaches infinity, the exponent and a fortiori the order of $\Sigma(N)$ go to infinity.*

The Riemann surface $X_0(N)$ is the group of complex points of a modular curve $X_0(N)_{\mathbf{Q}}$ defined over $\mathbf{Q}$. Therefore, $J_0(N)$ is naturally defined over $\mathbf{Q}$ and the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, where $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, acts on the group of torsion points of $J_0(N)$. It acts, in particular, on the Shimura subgroup $\Sigma(N)$. We determine this action in §4, and obtain:

THEOREM 2 .— *Let e be the exponent of the group $\Sigma(N)$ (see* cor. 2 *of* thm. 1*). The smallest common field of definition of the points of $\Sigma(N)$ is the cyclotomic field $\mathbf{Q}(\mu_e)$. The Galois group $\mathrm{Gal}(\mathbf{Q}(\mu_e)/\mathbf{Q})$ acts on $\Sigma(N)$ via the cyclotomic character $\mathrm{Gal}(\mathbf{Q}(\mu_e)/\mathbf{Q}) \to (\mathbf{Z}/e\mathbf{Z})^\times$.*

COROLLARY 1 .— *A point x of $\Sigma(N)$ is rational over $\mathbf{Q}$ if and only if we have $2x = 0$. The number of those points is $2^{\mathrm{Card}(P)+\epsilon}$, where P is the set of odd primes dividing N and ϵ is given by*

$$\epsilon = \begin{cases} -1 & \textit{if } 4 \not| N \textit{ and there exists } p \in P,\ p \not\equiv 1 \bmod 8; \\ -1 & \textit{if } 4|N,\ 8 \not| N \textit{ and there exists } p \in P,\ p \not\equiv 1 \bmod 4; \\ 1 & \textit{if } 32|N; \\ 0 & \textit{otherwise.} \end{cases}$$

COROLLARY 2 .— *The only integers N for which all points of $\Sigma(N)$ are rational over $\mathbf{Q}$ are:*

(i) those for which $\Sigma(N)$ is of order 1, listed in cor. 3 *of* thm. 1*;*

(ii) the integers 20, 21, 24, 32, 48, 64, 72, 100, 144 and 147, for which $\Sigma(N)$ is of order 2;

(iii) the integers 96, 192, 288 and 576, for which $\Sigma(N)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$.

To each divisor N_1 of N, such that N_1 is prime to N/N_1, is associated an *Atkin-Lehner involution* w_{N_1} of $X_0(N)$: for the definition, see §5. The involutions $w_{N_1}{}^*$ and $(w_{N_1})_*$ of $J_0(N)$ deduced by Picard and Albanese functorialities respectively coincide. The behaviour of the Shimura subgroup of $J_0(N)$ under these maps is studied in §5. We obtain:

THEOREM 3 .— *The Shimura subgroup $\Sigma(N)$ of $J_0(N)$ is stable under $w_{N_1}{}^*$. Moreover, we have the commutative diagram*

$$\begin{array}{ccc} \Sigma(N) & \xrightarrow{\psi'} & \mathrm{Hom}((\mathbf{Z}/N\mathbf{Z})^\times, \mathbf{U}) \\ \alpha \downarrow & & {}^t\alpha' \downarrow \\ \Sigma(N) & \xrightarrow{\psi'} & \mathrm{Hom}((\mathbf{Z}/N\mathbf{Z})^\times, \mathbf{U}), \end{array} \tag{3}$$

where α is the map induced by $w_{N_1}{}^*$, ψ' is the canonical injection (2), and ${}^t\alpha'$ is the transpose of the involution $\alpha' : (\mathbf{Z}/N\mathbf{Z})^\times \to (\mathbf{Z}/N\mathbf{Z})^\times$ which coincides with $t \mapsto t^{-1}$ modulo N_1 and with the identity modulo N/N_1.

The following particular case of thm. 3 was previously obtained by K. Ribet ([5], lemma 1):

COROLLARY .— *The involution $w_N{}^*$ acts on the Shimura subgroup $\Sigma(N)$ by multiplication by -1.*

Let M be a divisor of N. For each divisor D of N/M, we have a holomorphic degeneracy map $v_D : X_0(N) \to X_0(M)$. It is the map deduced from the transformation $\tau \mapsto D\tau$ of $\overline{\mathcal{H}}$ by passing to the quotients; a modular definition of v_D is given in §6. By Picard and Albanese functorialities respectively, we get morphisms of abelian varieties

$$\begin{aligned} v_D{}^* : J_0(M) &\longrightarrow J_0(N), \\ (v_D)_* : J_0(N) &\longrightarrow J_0(M), \end{aligned} \tag{4}$$

the latter being the dual of the former. The behaviour of the Shimura subgroups under these maps is studied in §6. We obtain:

THEOREM 4 .— *We have $v_D{}^*(\Sigma(M)) \subseteq \Sigma(N)$. Moreover, we have the commutative diagram*

$$\begin{array}{ccc} \Sigma(M) & \longrightarrow & \mathrm{Hom}((\mathbf{Z}/M\mathbf{Z})^\times, \mathbf{U}) \\ \beta \downarrow & & {}^t\beta' \downarrow \\ \Sigma(N) & \longrightarrow & \mathrm{Hom}((\mathbf{Z}/N\mathbf{Z})^\times, \mathbf{U}), \end{array} \tag{5}$$

where β is the map induced by $v_D{}^*$, the horizontal arrows represent the canonical injections (2), and ${}^t\beta'$ is the transpose of the canonical surjection $\beta' : (\mathbf{Z}/N\mathbf{Z})^\times \to (\mathbf{Z}/M\mathbf{Z})^\times$.

THEOREM 5 .— *We have $(v_D)_*(\Sigma(N)) \subseteq \Sigma(M)$. Moreover, we have the commutative diagram*

$$\begin{array}{ccc} \Sigma(N) & \longrightarrow & \mathrm{Hom}((\mathbf{Z}/N\mathbf{Z})^\times, \mathbf{U}) \\ \delta \downarrow & & {}^t\delta' \downarrow \\ \Sigma(M) & \longrightarrow & \mathrm{Hom}((\mathbf{Z}/M\mathbf{Z})^\times, \mathbf{U}), \end{array} \tag{6}$$

where δ is the map induced by $(v_D)_*$, the horizontal arrows represent the canonical injections (2), and ${}^t\delta'$ is the transpose of the homomorphism