

BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 122, n° 2 (1994), p. 163-184

http://www.numdam.org/item?id=BSMF_1994__122_2_163_0

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ON MICROLOCAL b -FUNCTION

BY

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RÉSUMÉ. — Soit f un germe de fonction holomorphe en n variables. En utilisant des opérateurs différentiels microlocaux, on introduit la notion de b -fonction microlocale $\tilde{b}_f(s)$ de f , et on démontre que $(s+1)\tilde{b}_f(s)$ coïncide avec la b -fonction (i.e. le polynôme de Bernstein) de f . Soient R_f les racines de $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$ et $m_\alpha(f)$ la multiplicité de $\alpha \in R_f$. On démontre $R_f \subset [\alpha_f, n - \alpha_f]$ et $m_\alpha(f) \leq n - \alpha_f - \alpha + 1$ ($\leq n - 2\alpha_f + 1$). Le théorème de type Thom-Sebastiani pour b -fonction est aussi démontré sous une hypothèse raisonnable.

ABSTRACT. — Let f be a germ of holomorphic function of n variables. Using microlocal differential operators, we introduce the notion of microlocal b -function $\tilde{b}_f(s)$ of f , and show that $(s+1)\tilde{b}_f(s)$ coincides with the b -function (i.e. Bernstein polynomial) of f . Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$. Then we prove $R_f \subset [\alpha_f, n - \alpha_f]$ and $m_\alpha(f) \leq n - \alpha_f - \alpha + 1$ ($\leq n - 2\alpha_f + 1$). The Thom-Sebastiani type theorem for b -function is also proved under a reasonable hypothesis.

Introduction

Let f be a holomorphic function defined on a germ of complex manifold (X, x) . The b -function (i.e., Bernstein polynomial) $b_f(s)$ of f is defined by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.1) \quad b(s)f^s = Pf^{s+1} \quad \text{in } \mathcal{O}_{X,x}[f^{-1}][s]f^s$$

for $P \in \mathcal{D}_{X,x}[s]$. Let $\delta(t-f)$ denote the delta function on $X' := X \times \mathbb{C}$ with support $\{f=t\}$, where t is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t t$, f^s and $\delta(t-f)$ satisfy the same relation (see for example [8]). So f^s in (0.1) can be replaced by $\delta(t-f)$, and f^{s+1} by $t\delta(t-f)$. We define the

(*) Texte reçu le 21 février 1992, révisé le 6 décembre 1992.

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microlocal b -function $\tilde{b}_f(s)$ by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(0.2) \quad b(s)\delta(t-f) = P\partial_t^{-1}\delta(t-f) \quad \text{in } \mathcal{O}_{X,x}[\partial_t, \partial_t^{-1}]\delta(t-f)$$

for $P \in \mathcal{D}_{X,x}[\partial_t^{-1}, s]$. Here we can also allow for P a microdifferential operator [4], [6], [17] satisfying a condition on the degree of t and ∂_t (see (1.4)). We have :

PROPOSITION 0.3. — $b_f(s) = (s+1)\tilde{b}_f(s)$.

See (1.5). The microlocal b -function $\tilde{b}_f(s)$ is sometimes easier to treat than the b -function $b_f(s)$. Let R_f be the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, $m_\alpha(f)$ the multiplicity of $\alpha \in R_f$, and $n = \dim X$. Then, using the duality of filtered $\mathcal{D}$ -Modules [15] and the theory of Hodge Modules [12], we prove

THEOREM 0.4. — $R_f \subset [\alpha_f, n - \alpha_f]$.

THEOREM 0.5. — $m_\alpha(f) \leq n - \alpha_f - \alpha + 1 \quad (\leq n - 2\alpha_f + 1)$.

See (2.8), (2.10).

The estimate (0.4) is optimal because $\max R_f = n - \alpha_f$ in the quasi-homogeneous isolated singularity case. See also remark after (2.8) below. Note that $R_f \subset \mathbb{Q}$ and $\alpha_f > 0$ by [4], and (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (with $\delta_{\alpha,1}$ Kronecker's delta) which is shown in [9] as a corollary of the relation with Deligne's vanishing cycle sheaf $\varphi_f \mathbb{C}_X$ [2] (see also [5]). This relation implies for example that $\exp(2\pi i\alpha)$ for $\alpha \in R_f$ are the eigenvalues of the monodromy on $\varphi_f \mathbb{C}_X$. But $\varphi_f \mathbb{C}_X$ cannot be replaced with the reduced cohomology of a Milnor fiber at x as in the isolated singularity case, because we have to take the Milnor fibration at several points of $\text{Sing } f^{-1}(0)$ even when we consider the b -function of f at x . See (2.12) below.

Let T_u and T_s denote respectively the unipotent and semisimple part of the monodromy T on $\varphi_f \mathbb{C}_X$. Let $\varphi_f^\alpha \mathbb{C}_X = \text{Ker}(T_s - \exp(-2\pi i\alpha))$ (as a shifted perverse sheaf), and $N = \log T_u / 2\pi i$. In the proof of (0.5), we get also :

PROPOSITION 0.6. — We have $N^{r+1} = 0$ on $\varphi_f^\alpha \mathbb{C}_X$ for $\alpha \in [\alpha_f, \alpha_f + 1]$ and $r = [n - \alpha_f - \alpha]$. In particular, $N^{r+1} = 0$ on $\varphi_f \mathbb{C}_X$ for $r = [n - 2\alpha_f]$.

For the proof of (0.4)–(0.6), we use the filtration V (similar to that in [5], [9]) defined on the $\mathcal{D}_{X,x}[t, \partial_t, \partial_t^{-1}]$ -module $\tilde{\mathcal{B}}_f$ generated by the delta function $\delta(t-f)$. Note that (0.3) may be viewed as an extension

of Malgrange's result [8] to the nonisolated singularity case (see (1.7) below), and in the isolated singularity case, (0.4)–(0.6) can be deduced from results of [8], [19], [20] (and [18]) using an argument as in [14]. In the nondegenerate Newton boundary case [7], we get an estimate of α_f using the Newton polyhedron (see (3.3)). The idea of its proof is essentially same as [16].

Let g be a holomorphic function on a germ of complex manifold (Y, y) . Let $Z = X \times Y, z = (x, y)$, and $h = f + g \in \mathcal{O}_{Z, z}$. We define R_g, R_h as above. Then we have :

PROPOSITION 0.7. — $R_f + R_g \subset R_h + \mathbb{Z}_{\leq 0}, R_h \subset R_f + R_g + \mathbb{Z}_{\geq 0}$.

THEOREM 0.8. — Assume there is a holomorphic vector field ξ such that $\xi g = g$. Then we have $R_f + R_g = R_h$, and

$$m_\gamma(h) = \max_{\alpha+\beta=\gamma} \{m_\alpha(f) + m_\beta(g) - 1\}.$$

See (4.3)–(4.4). Here $\mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}_{\leq 0}$) is the set of nonnegative (or non-positive) integers. In the case where f and g have isolated singularities, (0.7)–(0.8) can be easily deduced from results of MALGRANGE [8], [10] (see (4.6) below), and (0.8) was first obtained by [21] in this case. Note that (0.8) is not true in general if the hypothesis is not satisfied. See (4.8) below.

1. Microlocal b -function

1.1. — Let X be a complex manifold of pure dimension n , and $x \in X$. Let $\mathcal{O} = \mathcal{O}_{X, x}, \mathcal{D} = \mathcal{D}_{X, x}$. We define rings $\mathcal{R}, \tilde{\mathcal{R}}$ by

$$(1.1.1) \quad \mathcal{R} = \mathcal{D}[t, \partial_t], \quad \tilde{\mathcal{R}} = \mathcal{D}[t, \partial_t, \partial_t^{-1}],$$

where t, ∂_t satisfy the relation $\partial_t t - t \partial_t = 1$, and $\mathcal{D}[t, \partial_t] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[t, \partial_t]$, etc. We define the filtration V on $\mathcal{R}, \tilde{\mathcal{R}}$ by the differences of the degrees of t and ∂_t :

$$(1.1.2) \quad V^p \mathcal{R} = \sum_{i-j \geq p} \mathcal{D} t^i \partial_t^j \quad (\text{same for } \tilde{\mathcal{R}}).$$

Then we have :

$$(1.1.3) \quad \begin{cases} V^p \mathcal{R} = t^p V^0 \mathcal{R} = V^0 \mathcal{R} t^p & (p > 0), \\ V^{-p} \mathcal{R} = \sum_{0 \leq j \leq p} \partial_t^j V^0 \mathcal{R} = \sum_{0 \leq j \leq p} V^0 \mathcal{R} \partial_t^j & (p > 0), \\ V^p \tilde{\mathcal{R}} = \partial_t^{-p} V^0 \tilde{\mathcal{R}} = V^0 \tilde{\mathcal{R}} \partial_t^{-p}. \end{cases}$$

1.2. — Let $f \in \mathcal{O}$ such that $f(0) = 0$ and $f \neq 0$. Let

$$(1.2.1) \quad \mathcal{B}_f = \mathcal{O}[\partial_t]\delta(t-f), \quad \tilde{\mathcal{B}}_f = \mathcal{O}[\partial_t, \partial_t^{-1}]\delta(t-f),$$

where $\mathcal{O}[\partial_t]\delta(t-f)$ is a free module of rank one over $\mathcal{O}[\partial_t]$ ($= \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$) with a basis $\delta(t-f)$ (similarly for $\tilde{\mathcal{B}}_f$). Here $\delta(t-f)$ denotes the delta function supported on $\{f=t\}$ (see remark below). We have a structure of $\mathcal{R}$ -module and $\tilde{\mathcal{R}}$ -module on $\mathcal{B}_f$ and $\tilde{\mathcal{B}}_f$ respectively by

$$(1.2.2) \quad \begin{cases} \xi(a\partial_t^i\delta(t-f)) = (\xi a)\partial_t^i\delta(t-f) - (\xi f)a\partial_t^{i+1}\delta(t-f), \\ t(a\partial_t^i\delta(t-f)) = fa\partial_t^i\delta(t-f) - ia\partial_t^{i-1}\delta(t-f) \end{cases}$$

for $a \in \mathcal{O}$ and $\xi \in \Theta_{X,x}$. We define a decreasing filtration G on $\mathcal{B}_f, \tilde{\mathcal{B}}_f$ by

$$(1.2.3) \quad G^p\mathcal{B}_f = V^p\mathcal{R}\delta(t-f), \quad G^p\tilde{\mathcal{B}}_f = V^p\tilde{\mathcal{R}}\delta(t-f),$$

and an increasing filtration F by

$$(1.2.4) \quad F_p\mathcal{B}_f = \bigoplus_{0 \leq i \leq p} \mathcal{O}\partial_t^i\delta(t-f), \quad F_p\tilde{\mathcal{B}}_f = \bigoplus_{i \leq p} \mathcal{O}\partial_t^i\delta(t-f)$$

Then we have :

$$(1.2.5) \quad \partial_t^i : G^p\tilde{\mathcal{B}}_f \xrightarrow{\sim} G^{p-i}\tilde{\mathcal{B}}_f, \quad \partial_t^i : F_p\tilde{\mathcal{B}}_f \xrightarrow{\sim} F_{p+i}\tilde{\mathcal{B}}_f,$$

$$(1.2.6) \quad \mathcal{D}_{X,x}[s](F_p\tilde{\mathcal{B}}_f) \subset G^{-p}\tilde{\mathcal{B}}_f.$$

Remark. — The $\mathcal{R}$ -module $\mathcal{B}_f$ is identified with the germ at $(x,0)$ of the direct image of $\mathcal{O}_X$ as $\mathcal{D}$ -Module by the closed embedding i_f defined by the graph of f , where t is identified with the coordinate of $\mathbb{C}$. See [4] and [17].

1.3 Definition. — The b -function $b_f(s)$ (resp. *microlocal b -function* $\tilde{b}_f(s)$) is defined by the minimal polynomial of the action of $s := -\partial_t t$ on $\mathrm{Gr}_G^0\mathcal{B}_f$ (resp. $\mathrm{Gr}_G^0\tilde{\mathcal{B}}_f$).

REMARK. — Since $\mathrm{Gr}_V^0\mathcal{R} = \mathrm{Gr}_V^0\tilde{\mathcal{R}} = \mathcal{D}[s]$, $b_f(s)$ (resp. $\tilde{b}_f(s)$) is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$(1.3.1) \quad b(s)\delta(t-f) = P\delta(t-f)$$