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SIGN CHANGES OF THE PARTIAL SUMS OF A RANDOM MULTIPLICATIVE FUNCTION III: AVERAGE

by Marco Aymone

Abstract. — Let $V(x)$ be the number of sign changes of the partial sums up to x, say $M_f(x)$, of a Rademacher random multiplicative function f. We prove that the averaged value of $V(x)$ is at least $\gg (\log x)(\log\log x)^{-1/2-\epsilon}$. Our new method applies for the counting of sign changes of the partial sums of a system of orthogonal random variables having variance 1 under additional hypotheses on the moments of these partial sums. In particular, we extend to larger classes of dependencies an old result of Erdős and Hunt on sign changes of partial sums of i.i.d. random variables. In the arithmetic case, the main input in our method is the "*linearity*" phase in $1 \leq q \leq 1.9$ of the quantity $\log \mathbb{E}|M_f(x)|^q$, provided by the Harper's *better than square-root cancellation* phenomenon for small moments of $M_f(x)$.

Résumé (*Changements de signes de la fonction sommatoire d'une fonction multiplicative aléatoire III : moyenne*). — Soit $V(x)$ le nombre de changements de signes jusqu'à x de la fonction sommatoire $M_f(x)$, d'une fonction multiplicative aléatoire de Rademacher f. Nous prouvons que la valeur moyenne de $V(x)$ est au moins égale à $\gg (\log x)(\log\log x)^{-1/2-\epsilon}$. Notre nouvelle méthode s'applique plus généralement au dénombrement des changements de signe des fonctions sommatoires d'un système de variables aléatoires orthogonales de variance 1 sous une hypothèse supplémentaire sur les moments de ces fonctions. En particulier, nous étendons à une classe plus large un ancien résultat d'Erdős et Hunt sur les changements de signe des fonctions sommatoires de variables aléatoires i.i.d. Dans le cas arithmétique, la donnée principale de notre méthode est la phase de « linéarité » lorsque $1 \leq q \leq 1,9$ de la quantité $\log \mathbb{E}|M_f(x)|^q$, fournie par le phénomène « *plus de compensations que la racine carré* » de Harper pour les petits moments de $M_f(x)$.

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1. Introduction

1.1. Main result and background. — In a recent paper by Heap, Zhao, and the author [2], a proof was exhibited for the claim that the arithmetic random walk performed by the partial sums of a Rademacher random multiplicative function f, say $M_f(x) := \sum_{n\leq x} f(n)$, visits the origin an infinite number of times, almost surely (see also an extension of this result for weighted partial sums by the author [1]). The method presented there was used by Geis and Hiary [4] to give an almost sure lower bound for the number of sign changes $V(x)$ of $M_f(u)$ in the interval $u \in [1, x]$. They showed that

$$V(x) \gg_\delta (\log\log\log x)^{1/2-\delta}, \ a.s., \text{ for any } \delta > 0.$$

In a very recent update in a preprint by Klurman, Lamzouri, and Munsch [8], they showed that

$$V(x) \gg \frac{\log\log x}{\log\log\log\log x}, \ a.s.$$

In this third of a series of papers on sign changes of the partial sums of a random multiplicative function, we are interested in the average of $V(x)$.

Corollary 1.1. — *Let $V(x)$ be the number of sign changes of $M_f(u)$ in the interval $u \in [1, x]$. Then there exists a constant $\kappa > 0$, such that for each fixed $0 < \epsilon < 1/100$, for all $x \geq x_0 = x_0(\epsilon)$,*

$$\mathbb{E}V(x) \geq \kappa \frac{\log x}{(\log\log x)^{1/2+\epsilon}}.$$

In a series of two papers by Kaczorowski and Pintz [6] and [7], the best result known to date for the Mertens function was proved: there is at least a constant $c > 0$ times $\log x$ sign changes of $\sum_{n\leq u} \mu(n)$ in the interval $[1, x]$. Thus our corollary above shows a similar result between random multiplicative functions with what is known[1] for the Möbius function.

Remark 1.1. — As it is common in *Measure Theory*, the average behavior can be, most of the time, very different from the almost sure behavior. In this sense, we are unsure whether our Corollary 1.1 would lead to improvements in the almost sure result of Klurman, Lamzouri, and Munsch [8].

We obtain our Corollary 1.1 as a consequence of the following local result for sign changes:

Theorem 1.1. — *Let $N \in \mathbb{N}$ and $x \geq 1$ be sufficiently large and such that $N = o(\log x)$ and, for fixed $0 < \epsilon < 1/1000$, $\log\log x \ll N^{2-\epsilon}$. Then the*

1. For the i.i.d Rademacher random walk, we have, on average, $\asymp \sqrt{x}$ sign changes of this walk up to time x. It seems plausible to speculate something similar for the "Möbius walk" and for Rademacher random multiplicative functions.

probability that $M_f(u)$ has at least one sign change in the interval $u \in [x, e^N x]$ is at least an absolute constant $\kappa > 0$ for all x sufficiently large.

1.2. A more general Theorem and examples. — Our method has two main inputs. The first is the orthogonality of the random variables $(f(n))_n$. Under this orthogonality, it is easy to show that the correlation between $M_f(e^n x)$ and $M_f(e^m x)$ is exponentially small on the quantity $|m-n|$.

The second main input is the remarkable Harper's *better than square-root cancellation* [5] for the moments $\mathbb{E}|M_f(x)|^q$ for $1 \leq q < 2$. Actually, the better than the square-root cancellation is not, a priori, the key feature to prove our results, but the following linearity in q:

$$\log \mathbb{E}|M_f(x)|^q \asymp \frac{q}{2} \log\left(\frac{x}{(1+(1-q/2)\sqrt{\log\log x})}\right).$$

Our method allows us to prove the following result:

Theorem 1.2. — *Let $(X_n)_n$ be orthogonal random variables such that $\mathbb{E}X_n = 0$ and $\mathbb{E}X_n^2 = 1$ for all n. Let $M(u) := \sum_{n\leq u} X_n$ denote the partial sums of X_n. Assume that for some $1 \leq q_1 < q_2 \leq 2$, as $x \to \infty$, the L^{q_1} and L^{q_2} norm satisfy the linearity condition $\|M(x)\|_{q_1} \asymp \|M(x)\|_{q_2}$. Assume further that*

$$\|M(x)\|_{q_1} \asymp \frac{\sqrt{x}}{\psi(x)}$$

for some continuous and non-decreasing function $\psi : [1,\infty) \to [1,\infty)$. Further, assume that for $n = o(\log x)$,

$$\psi(e^n x) = \psi(x)\left(1 + O\left(\frac{n}{\log x}\right)\right).$$

If $\psi(x)^{2+\epsilon} \ll N = o(\log x)$ for some small fixed $0 < \epsilon < 1/100$, then the probability that $M(u)$ has at least one sign change in the interval $u \in [x, e^N x]$ is at least an absolute constant κ if x is sufficiently large.

1.2.1. *Examples.* —

Example 1.1. — If $\psi(x)$ can be taken to be a constant in Theorem 1.2, then the number of sign changes $V(x)$ of the partial sums $M(u)$ in the interval $u \in [1, x]$ is such that

$$\mathbb{E}V(x) \gg \log x.$$

Concrete examples of this can be achieved via *Martingale Theory* in Probability. Indeed, let $(X_n)_n$ be a Martingale difference, that is, a sequence of random variables with the following properties:

- $\mathbb{E}|X_n| < \infty$, for all n;
- there is a sequence of filtrations $(\mathcal{F}_n)_n$ such that X_n is $\mathcal{F}_n$-measurable;
- $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = 0$ for all $n \geq 1$.

The tower property for conditional expectation guarantees that the random variables (X_n) are orthogonal and $\mathbb{E}X_n = 0$, for all n. Moreover, we have the Burkholder's inequality (see the book of Shiryaev [9] p. 499): for each $q > 1$, there are constants $c_q, C_q > 0$ such that

$$c_q \left\| \sqrt{\sum_{n\leq x} X_n^2} \right\|_q \leq \|M(x)\|_q \leq C_q \left\| \sqrt{\sum_{n\leq x} X_n^2} \right\|_q .$$

So if we further assume that for some positive constants A and B, $A \leq |X_n| \leq B$ for all n, then the hypotheses of Theorem 1.2 are satisfied with $\psi(x) = \mathit{constant}$. Other subsets of hypotheses on $(X_n)_n$ could lead to same conclusions.

Example 1.2. — Theorem 1 of Erdős and Hunt [3] states that if $(X_n)_n$ are symmetric and i.i.d. random variables, then

$$\mathbb{E}V(x) \geq \frac{1}{2}\log x + O(1).$$

Our Theorem 1.2, in particular, partially recovers this result of Erdős and Hunt. We say partially, because in [3] they do not need any other assumption on the moments of $(X_n)_n$, while our Theorem 1.2 demands the existence of two moments that match their order of magnitude. On the other hand, we do not need the symmetry hypothesis, and as our previous example shows, our method can be used in other settings of dependencies among $(X_n)_n$.

Therefore, if we in addition assume that $(X_n)_n$ are i.i.d. and $\mathbb{E}X_1^{2+\epsilon} < \infty$, then we can use the Burkholder's inequality (or the Marcinkiewicz–Zygmund's inequality) of the previous example to guarantee that the first moment has the same order of magnitude as the second moment, via an interpolation-norm argument. With this we conclude that

$$\mathbb{E}V(x) \gg \log x.$$

Example 1.3 ($\mathcal{B}_2$ or Sidon sets). — A set $S = \{n_1, n_2, \ldots\}$ of positive integers is called $\mathcal{B}_2$ or Sidon if there exists a constant $C > 0$ such that for all $n \geq 1$, the equation

$$n = n_j \pm n_k$$

has at most C solutions with n_j and n_k in S. By letting U be a random variable with uniform distribution over $[0, 2\pi]$ and $M(x)$ the cosine polynomial

$$M(x) := \sum_{k\leq x} \cos(n_k U),$$

an old result of Sidon [10] states that

$$\mathbb{E}M(x)^4 \asymp (\mathbb{E}M(x)^2)^2 \asymp x^2.$$

This implies that

$$\mathbb{E}|M(x)| \asymp (\mathbb{E}M(x)^2)^{1/2},$$

and hence the hypotheses of Theorem 1.2 are satisfied with $\psi(x)$ a constant. Thus in this case we also have that

$$\mathbb{E}V(x) \gg \log x.$$

Example 1.4. — Our method is flexible enough to hold other situations where $\mathbb{E}X_n^2 \gg 1$ does not necessarily apply. For instance, the interested reader can outline the details and prove, by using our method, that in the case $X_n = r_n/\sqrt{n}$, where $(r_n)_n$ are i.i.d. Rademacher random variables, that

$$\mathbb{E}V(x) \gg \log\log x.$$

2. Preliminaries

2.1. Notation. — In this subsection we provide a summary of all recurrent notation used in this paper. We hope that it may be useful to the reader if they become overloaded with the large amount of notation used here.

2.1.1. *Letters appearing throughout the text.* — We will let the letter p always represent a generic prime number, n represent a positive integer, and x and y be real variables used as the edge of an index of summation; q is reserved for the size of the moments of certain random variables. We let c_j represent positive auxiliary constants and Greek letters the ultimate constants. λ will be used as a threshold in the event that a random variable is above or beyond this threshold. The letter f represents our Rademacher random multiplicative function. $\mathbb{P}$ is the probability of an event.

2.1.2. *Asymptotic notation.* — We use the standard Vinogradov notation $f(x) \ll g(x)$ or Landau's $f(x) = O(g(x))$ whenever there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$ for all x in a set of parameters. When not specified, this set of parameters it is an infinite interval (a, ∞) for sufficiently large $a > 0$. Sometimes is convenient to indicate the dependence of this constant on other parameters. For this, we use both $\ll_\delta$ and O_δ to indicate that c may depend on δ. We say that $f \asymp g$ if both $f \ll g$ and $g \ll f$ are realized. The standard $f(x) = o(g(x))$ means that $f(x)/g(x) \to 0$ when $x \to a$, where a could be a complex number or $\pm\infty$.

2.2. Some estimates. — For $x \geq 1$, $N \in \mathbb{N}$ and $1 \leq q < 2$ define

$$\Lambda(N, x, q) = \sum_{n \leq N} \frac{1}{(1 + (1 - q/2)\sqrt{\log\log(e^n x)})^{q/2}}. \tag{1}$$