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AVERAGE DIFFUSION RATE OF EHRENFEST WIND-TREE BILLIARDS

BY SIMON BARAZER

ABSTRACT. — One of the versions of the wind-tree model of Boltzmann gas, suggested by Paul and Tatiana Ehrenfest more than a century ago, can be seen as a billiard in the plane endowed with $\mathbb{Z} \oplus \mathbb{Z}$-periodic rectangular obstacles. In the breakthrough paper by V. Delecroix, P. Hubert, and S. Lelievre, the authors proved that the diffusion rate of trajectories in such a billiard is equal to $\frac{2}{3}$. That is, the *maximal* distance from the origin achieved by a point of a typical trajectory on a segment of time $[0, t]$ grows roughly as $t^{\frac{2}{3}}$ for large t. Here $\frac{2}{3}$ is the Lyapunov exponent of the associated renormalizing dynamical system.

This pioneering result does not tell, however, whether trajectories spend most of the time close or far from the initial point. In the current paper, we prove that the *average* distance from the origin grows with the same rate of $t^{\frac{2}{3}}$. In plain terms, it means that trajectories mostly stay as far as possible from the initial point (though it is known that the wind-tree billiard is recurrent, so trajectories occasionally pass close to the initial point).

More generally, fundamental rigidity results from A. Eskin and M. Mirzakhani completed by certain genericity results of J. Chaika and A. Eskin imply that the diffusion rate of almost all flat geodesic rays on any $\mathbb{Z}^d$-cover of a closed translation surface S is given by a certain Lyapunov exponent of the Kontsevich–Zorich cocycle on the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of S. In this paper we prove that in this most general setting, the *average* and *maximum* diffusion rates coincide.

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Résumé (*Taux de diffusion moyen dans le modèle du vent dans les arbres*). — Une des versions du modèle du vent dans les arbres a été suggéré par Paul et Tatiana Ehrenfest plus d'un siècle auparavant; il peut être vu comme un billiard periodique dans le plan où des obstacles réctangulaires sont centrés sur le réseau $\mathbb{Z}\oplus\mathbb{Z}$. Dans le papier fondateur de V. Delecroix, P. Hubert et S. Lelievre,les auteurs ont prouvé que le coefficient de diffusion des trajectoires dans un tel billiard est egal à $\frac{2}{3}$. En d'autres termes, la distance *maximale* atteinte par une trajectoire typique sur l'intervalle de temps $[0, t]$ croit comme $t^{\frac{2}{3}}$ pour de grandes valeurs de t. Le coefficient $\frac{2}{3}$ est le coefficient de Lyapunov associé à un processus de renormalisation des trajectoires. Ce résultat ne précise pas si les trajectoires passent une grande partie du temps loin de la position initiale.

Dans cet article on prouve que la *moyenne* de la distance à l'origine grandit elle aussi comme $t^{\frac{2}{3}}$. Ce qui montre qu'une trajectoire passe une grande partie de son temps aussi loin que possible de la position initiale (même si il est connu que le flot du vent dans les arbres est récurent et occasionnellement, les tractoires passent près du point initial).

Plus généralement, des résultats profonds de rigidité prouvés par A. Eskin et M. Mirzakhani, complétés par des arguments de généricité de J. Chaika et A. Eskin impliquent que le taut de diffusion d'une trajectoire géodésique sur un revêtement abélien d'une surface de translation compacte est donné par un éxposant de Lyapunov du cocycle de Kontsevich-Zorich associé à la plus petite sous variété affine $\mathrm{SL}_2(\mathbb{R})$ invariante et contenant S. Dans cet article, on prouve dans ce cadre plus général que les taux de diffusion *maximal* et *moyen* coincident.

1. Introduction

The wind-tree model is an infinite $\mathbb{Z}^2$-periodic billiard in the plane $\mathbb{R}^2$, where identical rectangular obstacles are placed in such way that the corresponding sides are aligned with the horizontal and vertical axes and the centers of rectangles are located at the lattice points (see figure 1.1). More formally, for every couple of real parameters $(a, b) \in (0, 1)^2$, the wind-tree table $T(a, b)$ is defined as

$$T(a,b) = \mathbb{R}^2 \setminus \bigsqcup_{(m,n)\in\mathbb{Z}^2} \left(m - \frac{a}{2}, m + \frac{a}{2}\right) \times \left(n - \frac{b}{2}, n + \frac{b}{2}\right).$$

It is assumed that reflections on the sides of the rectangles are regular and the speed of all trajectories is constant and equal to one. A version of this model was introduced by Tatiana and Paul Erhenfest more than a century ago (see [5]). It has been studied more recently using the tools of translation surfaces theory. Athur Avila and Pascal Hubert proved in [1] that the trajectories in the wind-tree are recurrent. Later, Vincent Delecroix, Pascal Hubert, and Samuel Lelièvre proved in [3] that the diffusion rate of a typical trajectory is equal to $\frac{2}{3}$. If $\gamma_t(x, \theta)$ is the wind-tree flow in the direction θ starting from $x \in T(a, b)$, they proved the following theorem:

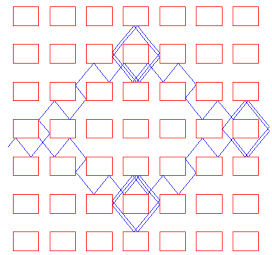

FIGURE 1.1. Trajectory in a Wind-Tree

THEOREM 1.1 (Delcroix, Hubert, Lelievre, 2014). — *For any* $(a, b) \in (0, 1)^2$, *for almost every* θ *(depending on* (a, b)*) and for all* $x \in T(a, b)$ *having an infinite future orbit, the following relation holds:*

$$\limsup_{t \to +\infty} \frac{\log d(x, \gamma_t(x, \theta))}{\log t} = \frac{2}{3}\,.$$

Anton Zorich and Vincent Delecroix generalized this result to a large class of wind-tree billiards with more complicated obstacles [4]. Charles Fougeron studied orbit closures of flat surfaces associated to even more general wind-tree billiards in [10]. In particular, he identified the Lyapunov exponent of the Kontsevich–Zorich cocycle that governs the associated diffusion rate. In a different direction, Krzysztof Fraczek and Corinna Ulcigrai showed in [11] that the directional flow for almost all wind-tree billiards is non-ergodic.

Note that the diffusion rate $\frac{2}{3}$ is defined in the original paper [3] of Delecroix–Hubert–Lelièvre as the *maximal* distance from the origin achieved by a point of a typical trajectory on a large interval of time. In the present paper, we study the diffusion *on average*. In particular, we prove the following theorem:

THEOREM 1.2. — *For any* $(a, b) \in (0, 1)^2$, *for almost every* θ *(depending on* (a, b)*) and for all* $x \in T(a, b)$ *having an infinite future orbit, the following relation holds:*

$$\lim_{T \to +\infty} \frac{\log \frac{1}{T} \int_0^T d(x, \gamma_t(x, \theta))dt}{\log T} = \frac{2}{3}\,.$$

The strategy of the proof is similar to the proof of Theorem 1.1. The first step is standard: following the Katok–Zemliakov construction ([21]; see also [17]), the authors of [3] construct a translation surface $X_\infty(a, b)$ without boundary from four copies of the wind-tree table $T(a, b)$. By construction, the billiard flow in $T(a, b)$ is conjugate to the geodesic flow on the translation surface $X_\infty(a, b)$. The fact that the wind-tree table is $\mathbb{Z}^2$-periodic implies that the surface $X_\infty(a, b)$ is a $\mathbb{Z}^2$-cover over a compact translation surface $X(a, b)$. In this way, we reduce the problem to a particular case of the study of geodesics on $\mathbb{Z}^d$-covers of translation surfaces.

We call $\pi : \tilde{S} \to S$ a *lattice cover* if $\tilde{S}$ is a connected $\mathbb{Z}^d$-cover of a compact translation surface S. We will see that such a cover is defined by a cohomology class f in the first cohomology $H^1(S, \mathbb{Z}^d)$. As in [3], the diffusion of a billiard trajectory can be studied through the intersection pairing of f with geodesic segments on S completed to closed paths in a natural way. We denote by $\tilde{\phi}_t(\tilde{x}, \theta), \phi_t(x, \theta)$ the geodesic flows on $\tilde{S}$ and S, respectively. Let $d(\tilde{x}, \tilde{y})$ be the distance between points $\tilde{x}$ and $\tilde{y}$ on the translation surface $\tilde{S}$. We prove the following theorem:

Theorem 1.3. — *Let $\pi : \tilde{S} \to S$ be a lattice cover; let f be the associated cohomology class in $H^1(S, \mathbb{Z}^d)$. For every $\tilde{x} \in \tilde{S}$ and almost every θ we have*

$$\lim_{T \to +\infty} \frac{\log \frac{1}{T} \int_0^T d(\tilde{x}, \tilde{\phi}_t(\tilde{x}, \theta)) dt}{\log T} = \Lambda(f),$$

where $\Lambda(f)$ is the Lyapunov exponent of f for the Kontsevich–Zorich cocycle and with respect to the affine invariant measure supported by the orbit closure of S for the action of $SL_2(\mathbb{R})$.

The proof of Theorem 1.3 uses tools from dynamics on the moduli spaces of translation surfaces. The corresponding technique was developed by A. Zorich in [23] and by G. Forni in [9], and it was applied in [3]. Let $\pi : \tilde{S} \to S$ be a lattice cover, and let f be the associated cohomology class. Let $(\tilde{x}, x)$ be a pair of points in $\tilde{S}$ and in S, respectively, such that $\pi(\tilde{x}) = x$. To study the drift of a trajectory starting from $\tilde{x}$, we project the trajectory to the base S. We consider a small horizontal interval I on S with one of the endpoints at x and the first return map of the vertical geodesic flow to I. The first return map is generally an interval exchange transformation; the vertical flow on S can be represented by Veech's zippered rectangles construction as a suspension flow over this interval exchange transformation. Consider a piece of trajectory starting at x and going up to the Nth visit to the interval I. Joining the endpoints of such a piece of trajectory along I, we get a cycle $C_N(S, x)$ in the first homology of S. The drift of the trajectory in $\tilde{S}$ is given by the intersection pairing of this cycle with the cohomology class f:

$$\langle f, C_N(S, x) \rangle . \tag{1.1}$$

Following Zorich and Forni, we use a renormalization procedure given by the Teichmüller flow on the moduli space of translation surfaces to study the behavior of the above quantity for large N. The Teichmüller flow g_s contracts vertical trajectories, so for an appropriate choice of time $s(N)$, cycles $C_N(S, x)$ representing long trajectories of ϕ_t (and thus having a large norm) are mapped to cycles on the new surface $g_s \cdot S$ having a norm comparable to 1. The growth of expression in equation 1.1 is then given by the Kontsevich–Zorich cocycle governed by the Gauss–Manin connection on the Hodge bundle. If we fix a cross section for the Teichmüller flow in the moduli space of translation surfaces, we obtain a sequence of flat surfaces S_n in the cross section such that

$$g_{s_n} \cdot S = \Gamma_n \cdot S_n \,,$$

where Γ_n is an element of the mapping class group and s_n is the nth return time to the cross section in the moduli space. We have the relation $C_{N_n}(S) = (\Gamma_n)_* C_l(S_n)$, where N_n is a sequence that grows to ∞ and l_n is bounded. Long cycles in S correspond to short cycles in S_n. Then, the behavior of $\langle f, C_{N_n}(S)\rangle$ for large n can be related to the behavior of the sequence $\Gamma_n^* f$, which is given by Oseledets' theorem.

In section 2 we recall some classical results in the theory of translation surfaces. That is, we recall the definition of moduli spaces of translation surfaces, the action of $\mathrm{SL}_2(\mathbb{R})$, and define the Kontsevich–Zorich cocycle on the Hodge bundle. We also recall fundamental theorems on dynamics on moduli spaces of flat surfaces.

In section 3 we prove Theorem 1.3. Using Veech's zippered rectangles construction presented in Paragraph 3, we relate the statement of the Theorem 1.3 to the behavior of the sum

$$\sum_{k \le N} \langle f, C_k(S, x)\rangle$$

for large N. We define a convenient cross-section for the Teichmüller flow in paragraph 3. In paragraph 3 we use this cross section and the renormalization procedure to construct trajectories with good excursion rates by applying Oseledets' theorem. In Paragraph 3 we establish the upper bound in Theorem 1.3 using Oseledets' theorem. To obtain the lower bound, we use the trajectories constructed in Paragraph 3 and a uniform upper bound to control the diffusion near these excursions. Finally, a proof of a technical lemma (Lemma 3.3) is placed in an appendix.

2. Background facts and constructions

The present paper uses tools from the theory of compact flat surfaces. We refer to the surveys of A. Zorich [24] and J.C. Yoccoz [20] for an elementary introduction to the subject and to a survey [9] of G. Forni and C. Matheus for more details. In this section, we introduce the material which will be used in the proof of Theorems 1.2 and 1.3 announced in the introduction. We start by presenting some basic facts on flat surfaces and lattice coverings. We also define the Teichmüller and moduli spaces of flat surfaces and give recent results related to the dynamics of the action of $\mathrm{SL}_2(\mathbb{R})$. Finally, we recall Veech's zippered rectangles construction, which we use in the study of the geodesic flow of a flat surface.