Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

CRYSTALLINE COHOMOLOGY OF RIGID ANALYTIC SPACES

Haoyang Guo

Tome 153
Fascicule 4

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 831-948

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 4, tome 153, décembre 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (4), 2025, p. 831-948

CRYSTALLINE COHOMOLOGY OF RIGID ANALYTIC SPACES

BY HAOYANG GUO

ABSTRACT. — In this article, we introduce the infinitesimal cohomology for rigid analytic spaces that are not necessarily smooth, with coefficients in a p-adic field or in Fontaine's de Rham period ring $\mathrm{B}_{\mathrm{dR}}^{+}$.

RÉSUMÉ (*Cohomologie cristalline des espaces*). — Dans cet article, nous introduisons la cohomologie infinitésimale pour les espaces analytiques rigides qui ne sont pas nécessairement lisses, avec des coefficients dans un corps p-adique ou dans l'anneau des périodes de de Rham de Fontaine $\mathrm{B}_{\mathrm{dR}}^{+}$.

1. Introduction

1.1. Background. — Let X be a complex algebraic variety. Attached to the set of $\mathbb{C}$-points of X, there is a natural analytic structure which makes $X(\mathbb{C})$ a complex analytic space. This allows us to obtain a topological invariant of X via singular cohomology $\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}), \mathbb{C})$, which is computed transcendentally.

Texte reçu le 5 janvier 2024, modifié le 1er mai 2024, accepté le 24 septembre 2024.

HAOYANG GUO, The University of Chicago, 5734 S University Ave, Chicago, IL 60637, United States • *E-mail :* hyguo@umich.edu

Mathematical subject classification (2010). — 14G22, 14F30.

Key words and phrases. — Rigid analytic geometry, p-adic cohomology, singularity.

The project is part of the author's thesis. It is partially funded by a Graduate Student Summer Fellowship from the Department of Mathematics, University of Michigan, and by NSF grant DMS 1801689 through Bhargav Bhatt. During the final revision process, the author is supported by the University of Chicago.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2025/831/$ 5.00

doi:10.24033/bsmf.2912

As the topological space $X(\mathbb{C})$ comes from an algebraic variety, it is natural to ask whether we could compute this singular cohomology algebraically. When X is a smooth algebraic variety over $\mathbb{C}$, it is a result of Grothendieck ([19]) that singular cohomology is isomorphic to algebraic de Rham cohomology. Thus, there exists a natural isomorphism

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X, \Omega^\bullet_{X/\mathbb{C}}),$$

where $\Omega^i_{X/\mathbb{C}}$ is the sheaf of the i-th algebraic Kähler differentials over the variety X, and $\Omega^\bullet_{X/\mathbb{C}}$ is the algebraic de Rham complex. As a consequence, we get a purely algebraic way to compute singular cohomology group.

However, if X is non-smooth, the cohomology of the usual algebraic de Rham complex may fail to compute singular cohomology of $X(\mathbb{C})$ (c.f. [2, Example 4.6]). To get the correct answer, in particular to get an algebraic cohomology theory which computes singular cohomology, there are several methods modifying algebraic de Rham cohomology in the non-smooth setting:

(1) In [24], Hartshorne discovered that if X admits a closed immersion into a smooth variety Y, then the formal completion $\widehat{\Omega^\bullet_{Y/\mathbb{C}}}$ of the de Rham complex $\Omega^\bullet_{Y/\mathbb{C}}$ along $X \to Y$ computes singular cohomology. Precisely, there exists the following isomorphism:

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X, \widehat{\Omega^\bullet_{Y/\mathbb{C}}}).$$

The result was obtained independently by Deligne (unpublished) and by Herrera–Lieberman [25].

In the general case when X is not necessarily embeddable, there exists a ringed *infinitesimal site* $(X/\mathbb{C}_{\mathrm{inf}}, \mathcal{O}_{X/\mathbb{C}})$ (or the crystalline site in characteristic zero) introduced by Grothendieck [20]. It can be shown that its cohomology $\mathrm{H}^i(X/\mathbb{C}_{\mathrm{inf}}, \mathcal{O}_{X/\mathbb{C}})$ coincides with $\mathrm{H}^i(X, \widehat{\Omega^\bullet_{Y/\mathbb{C}}})$ whenever $X \to Y$ is a closed immersion into a smooth variety as above. In particular, we obtain a conceptual cohomology theory that is independent of immersions. Moreover, the method allows us to compute cohomology with nontrivial coefficients, where we could replace $\mathcal{O}_{X/\mathbb{C}}$ by vector bundles with flat connections (or in other words, *crystals*).

(2) Extending the de Rham complex of smooth $\mathbb{C}$-algebras via simplicial resolutions, one obtains the *(Hodge-completed) derived de Rham complex*, first invented by Illusie [28]. To any scheme X over $\mathbb{C}$, we can associate a filtered derived algebra $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$. It was shown by Illusie in loc. cit. that the cohomology of the derived de Rham complex $\widehat{\mathrm{dR}}_{X/\mathbb{C}}$ is isomorphic to the Hartshorne's cohomology, assuming X is a local complete intersection. Later on, using Adams completion from the algebraic topology, Bhatt [6] shows that the comparison is true for any finite type scheme in characteristic zero, without the l.c.i condition. In particular, for an

arbitrary variety $X/\mathbb{C}$, we get the isomorphism

$$\mathrm{H}^i_{\mathrm{Sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \mathrm{H}^i(X, \widehat{\mathrm{dR}}_{X/\mathbb{C}}).$$

Here we mention that the first graded piece of $\widehat{\mathrm{dR}_{X/\mathbb{C}}}$ is the cotangent complex $\mathbb{L}_{X/\mathbb{C}}$ up to a shift, which plays an important role in the deformation theory of schemes. Moreover, similarly to the universal property of the algebraic de Rham complex, it could be shown that the derived de Rham complex is the initial object among all filtered (derived) algebras $\mathcal{A}$ over X that is equipped with a homomorphism $\mathcal{O}_X \to \mathrm{gr}^0\mathcal{A}$ ([37]).

(3) Another modification of the de Rham complex is called the *Deligne–Du Bois complex*, introduced by Deligne and studied by Du Bois ([12]). The Deligne–Du Bois complex is defined via the cohomological descent for resolution of singularities. More precisely, the Deligne–Du Bois complex for X is defined as the limit of the de Rham complex of X_n

$$R\lim_{[n]\in\Delta} Rf_{n*}\Omega^\bullet_{X_n/\mathbb{C}},$$

where $f_\bullet : X_\bullet \to X$ is a simplicial variety constructed using blowups at smooth nowhere dense centers, such that each X_n is smooth over $\mathbb{C}$. It could be shown that singular cohomology of X is isomorphic to the cohomology of the Deligne–Du Bois complex. Moreover, the Deligne–Du Bois complex admits a finite *Hodge–Deligne filtration* where each graded piece is a bounded complex of coherent sheaves in the derived category. The induced filtration on cohomology is the Hodge filtration for the underlying mixed Hodge structure. Furthermore, the Deligne–Du Bois complex together with its filtration also admits a site-theoretical interpretation via the h-topology, where the latter is introduced by Voevodsky in [42]. The theory of h-cohomology of X is studied for example in [26] and [32].

The above provides three algebraic methods computing singular cohomology of a complex variety that is not necessarily smooth. It is then natural to ask whether we have an analogous picture in the non-archimedean geometry. The goal of our article is to study the theory of cohomology for non-smooth rigid spaces in non-Archimedean geometry, analogous to the three modifications for complex algebraic varieties as above.

To start, let us fix some notations. Let K be a p-adic extension of $\mathbb{Q}_p$; i.e., K is a field that is complete with respect to a non-Archimedean valuation extending that of $\mathbb{Q}_p$. In the 1960s, Tate introduced the notion of the *rigid analytic space* that forms a natural p-adic analogue of the complex analytic space. Here, similarly to complex analytic spaces, examples of rigid analytic spaces include analytifications of algebraic varieties over K.

As a p-adic field is totally disconnected, singular cohomology of a rigid space over K is not an interesting object. However, we could still define a de Rham complex $\Omega^\bullet_{X/K}$ of X, where each $\Omega^i_{X/K}$ is the sheaf of Kähler differentials that are continuous under p-adic topology. When X is smooth and proper over K, it can be shown that the cohomology of its de Rham complex behaves very well: it lives within cohomological degrees $[0, 2\dim(X)]$, such that each cohomology group is a finite dimensional K-vector space. In particular, when X is the analytification of a smooth proper algebraic variety over K, the p-adic analytic de Rham cohomology of X and the algebraic de Rham cohomology of the variety match up, so we get the correct Betti numbers.

In the following, we consider general rigid spaces over K that may not be smooth.

1.2. Main results. — Let X be a rigid space over K. We introduce the *infinitesimal site* X/K_{inf}, defined on the category of all pairs of rigid spaces (U, T) for nil closed immersions $U \to T$ over K, such that U is an open subset in X. Here a collection of maps $\{(U_i, T_i) \to (U, T)\}$ is a covering in this site if $\{T_i \to T\}$ is an open covering for the rigid space T. The infinitesimal site X/K_{inf} naturally admits an *infinitesimal structure sheaf* $\mathcal{O}_{X/K}$, sending a thickening (U, T) to the ring of global sections $\mathcal{O}_T(T)$. Moreover, the infinitesimal structure sheaf admits a surjection onto $\mathcal{O}_X$, and the kernel ideal $\mathcal{J}_{X/K}$ defines a natural filtration on $\mathcal{O}_{X/K}$. The induced filtration on the cohomology of $\mathcal{O}_{X/K}$ is called the *infinitesimal filtration.*

Now we can formulate our first main result.

Theorem 1.2.1. — *There is a K-linear cohomology theory*

$$X \mapsto R\Gamma_{\mathrm{inf}}(X/K) := R\Gamma(X/K_{\mathrm{inf}}, \mathcal{O}_{X/K})$$

together with the filtration defined by $R\Gamma(X/K_{\mathrm{inf}}, \mathcal{J}^*_{X/K})$ for rigid spaces X over K, and it takes values in the filtered complete derived category of K-vector spaces. It satisfies the following properties:

(i) Explicit formula (Theorem 4.2.2): *Assume $X \to Y$ is a closed immersion into a smooth rigid space Y and J is the ideal sheaf defining X. Then $R\Gamma_{\mathrm{inf}}(X/K)$ is filtered isomorphic to the cohomology of the formal completion of the de Rham complex $\Omega^\bullet_{Y/K}$ along $X \to Y$:*

$$R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X, \widehat{\Omega^\bullet_{Y/K}}),$$

where the j-th filtration on the right side is $R\Gamma(X, \widehat{J^{j-\bullet}\Omega^\bullet_{Y/K}})$.

In particular, when X itself is smooth over k, the infinitesimal cohomology coincides with the de Rham cohomology with its Hodge filtration.

(ii) Derived de Rham comparison (Theorem 5.5.5): *There exists a natural filtered morphism from cohomology of the analytic derived de Rham complex to $R\Gamma_{\mathrm{inf}}(X/K)$:*

$$R\Gamma(X, \widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}) \longrightarrow R\Gamma_{\mathrm{inf}}(X/K).$$

The map induces an isomorphism on their underlying complexes and is a filtered isomorphism if X is a local complete intersection.

(iii) Éh comparison (Theorem 6.2.2): *The cohomology $R\Gamma_{\mathrm{inf}}(X/K)$ admits a natural filtered morphism to éh de Rham cohomology introduced in* [22]*, inducing an isomorphism on their underlying complexes:*

$$R\Gamma_{\mathrm{inf}}(X/K) \longrightarrow R\Gamma(X_{\mathrm{\acute{e}h}}, \Omega^{\bullet}_{\mathrm{\acute{e}h}}).$$

In particular, the underlying complex of $R\Gamma_{\mathrm{inf}}(X/K)$ satisfies the éh-hyperdescent for rigid spaces X over K.

(iv) Finiteness (Theorem 6.3.1): *When X is proper of dimension n over K, the underlying complex of $R\Gamma_{\mathrm{inf}}(X/K)$ is a perfect K-complex that lives in cohomological degree* $[0, 2\dim(X)]$.

(v) Base extension (Corollary 6.4.4): *Assume K_0 is a subfield of K and X is a proper rigid space over K_0. Then the natural base extension map is a filtered isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/K_0) \otimes_{K_0} K \longrightarrow R\Gamma_{\mathrm{inf}}(X_K/K).$$

(vi) Comparison with singular cohomology (Corollary 6.4.3): *Assume there exists an abstract isomorphism of fields $K \to \mathbb{C}$ and X is the analytification of a proper algebraic variety $\mathcal{X}$ over K. Then we have a filtered isomorphism*

$$\mathrm{H}^i_{\mathrm{inf}}(X/K) \otimes_K \mathbb{C} \simeq \mathrm{H}^i_{Sing}(\mathcal{X}(\mathbb{C}), \mathbb{C}),$$

where the latter is filtered by the algebraic infinitesimal filtration (cf. [6] *and* [24]*).*

Here the *underlying complex* of a filtered object is defined as the complex forgetting its filtration.

Remark 1.2.2 (Analytic derived de Rham complex). — In Theorem 1.2.1.(ii), the usual notion of the derived de Rham complex of Illusie is not suitable when we are working with rigid analytic spaces; instead, we modify the construction so that it is continuous under the p-adic topology. Our strategy is to first apply the (derived) p-adic completion to the algebraic derived de Rham complex for the rings of definitions over $\mathcal{O}_K$, then to consider the filtered completion of its generic fiber. This produces a filtered $\mathbb{E}_\infty$-algebra $\widehat{\mathrm{dR}}^{\mathrm{an}}_{X/K}$ in the derived (∞) category of sheaves of K-modules over X, whose graded pieces are wedge products of the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/K}$ introduced by Gabber–Ramero in [18, Section 7]. We refer readers to Subsection 5.3 for details.

This notion has been considered in [23], where Shizhang Li and the author show that applying at affinoid perfectoid algebras, the analytic derived de Rham complex can recover the de Rham period sheaves $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ over the pro-