Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 153
Fascicule 4

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 975-1001

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 4, tome 153, décembre 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (4), 2025, p. 975-1001

FIXED POINT THEOREMS FOR TOPOLOGICAL CONTRACTIONS AND THE HUTCHINSON OPERATOR

by Michał Morayne & Robert Rałowski

Abstract. — For a topological space X, a topological contraction on X is a closed mapping $f : X \to X$ such that for every open cover of X there is a positive integer n such that the image of the space X via the nth iteration of f is a subset of some element of the cover. Every topological contraction in a compact T_1 space has a unique fixed point. As in the case of metric spaces and the classical Banach fixed point theorem, this analogue of Banach's theorem is true not only in compact but also in complete (here in the sense of Čech) T_1 spaces. Kupka introduced the notion of a feeble topological contraction and proved the existence of a unique fixed point for such mappings with a closed graph without assuming completeness or compactness of the space considered. In particular, for Hausdorff spaces, this theorem implies the existence of a unique fixed point for continuous feeble contractions. We prove an analogue of this fact for closed feeble contractions for the first countable Hausdorff spaces. These theorems are applied to prove the existence of fixed points for mappings on compact subsets of linear spaces with weak topologies and for compact monoids. We also prove some fixed point results for T_1 locally Hausdorff spaces and, introduced here, peripherally Hausdorff spaces. An iterated function system on a topological space X, IFS, is a finite family of closed

Texte reçu le 4 août 2023, modifié le 10 janvier 2025, accepté le 18 février 2025.

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Mathematical subject classification (2010). — 54D10, 54D30, 54H25.

Key words and phrases. — Fixed point, T_1 space, compact space, contraction, Hutchinson operator.

The work of Robert Rałowski has been partially financed by grants 8211204601, 8211104160, MPK: 9120730000 from the Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology.

doi:10.24033/bsmf.2914

mappings from X into itself. It is contractive if for every open cover of X, for some positive integer n the image of X via each composition of n mappings from the IFS is contained in an element of the cover. We show that in T_1 compact topological spaces the Hutchinson operator of a contractive IFS may not be closed as a mapping into the hyperspace of closed subsets of the space. Nevertheless, the Hutchinson operator of a contractive IFS always has a unique fixed point.

RÉSUMÉ (*Théorèmes du point fixe pour des contractions topologiques et l'opérateur de Hutchinson*). — Pour un espace topologique X une contraction topologique sur X est une application $f : X \to X$ telle que pour tout recouvrement ouvert de X, il existe un nombre entier positif n tel que l'image de X par la n-ième iteration de f est un sous-ensemble d'un élément du recouvrement. Toute contraction topologique dans un espace compact T_1 possède un unique point fixe. Comme dans le cas des espaces métriques et du théorème classique de Banach, cet analogue du théorème de Banach est vrai non seulement pour les espaces T_1 compacts mais aussi pour les espaces T_1 complets. Kupka a introduit la notion de contraction topologique faible et il a démontré l'existence d'un unique point fixe pour de telles applications ayant le graph fermé sans supposer la compacité ou la complétude de l'espace considéré. En particulier, pour les espaces de Hausdorff ce théorème implique l'existence d'un point fixe pour les contractions faibles continues. Nous démontrons un analogue de ce fait pour les contractions faibles fermées pour les espaces de Hausdorff à bases dénombrable de voisinages. Ces théorèmes sont utilisés pour démontrer l'existence de points fixes d'applications sur des sous-ensembles d'espaces vectoriels munis de topologie faibles et sur les monoïds compacts. Nous démontrons aussi quelques résultats concernant de points fixes pour des espaces topologiques T_1 localement Hausdorff et pour les espaces périphériquements Hausdorff, introduits ici. Un système de fonctions itérés, SFI, dans un espace topologique X est une famille finie d'applications fermées de X dans lui-même. Un SFI est contractant si pout tout recouvrement ouvert de X il exist un nombre entier positif n tel que l'image de X par toute composition de n applications de SFI est contenu dans un élément du recouvrement. Nous démontrons que dans les espaces topologiques T_1 compacts l'opérateur de Hutchinson d'un SFI contractant, considéré comme une applications dans l'hyperespace des ensembles fermés de l'espace, peut ne pas être fermé. Neanmoins, l'opérateur de Hutchinson d'un SFI contractant a toujours un unique point fixe.

1. Introduction

The classical Banach contraction theorem states that Lipschitz mappings with Lipschitz constants smaller than 1 on a complete metric space, thus in particular on a compact metric space, to the same space have a unique fixed point. In [11] this theorem has been generalized to T_1 spaces, where the notion of a Lipschitz contraction, specific for metric spaces, was generalized by a notion of topological contraction. As Čech completeness, or actually a condition characterizing its original definition (e.g., [5]), extends the notion of completeness to topological spaces, one can ask the natural question of whether the above-mentioned analogue of Banach's fixed point theorem for T_1 compact spaces holds in T_1 Čech complete spaces. It turns out that it does, and this is the content of Theorem 3.2. One can also expect that in the case of T_2 spaces, some assumptions of an analogous theorem could be weakened. We present two such fixed point theorems for weaker notions of contractions, so called feeble topological contractions, with the assumptions that they are continuous (Theorem 4.1; this theorem is a direct corollary of a fairly general theorem of Kupka from [8]) or closed (Theorem 4.5); in this second case, we require that the space is first countable. These theorems are applied to obtain fixed point theorems for mappings on compact subsets of linear spaces with weak topologies.

In this article we consider special T_1 spaces which are more general than the Hausdorff spaces, the locally Hausdorff T_1 spaces. While the fixed point theorem for continuous feeble topological contractions extends to this class of spaces, the theorem for closed feeble topological contractions does not. We also introduce a class of T_1 peripherally Hausdorff spaces. For this class of spaces and for a stronger notion of contraction, a feeble+ topological contraction, we also obtain a fixed point theorem.

An iterated function system on a topological space X, an IFS, is a finite family of closed mappings from the space into itself (we do not assume their continuity here). Given an IFS, the Hutchinson operator generated by this IFS maps a set to the union of its images via all elements of the IFS. In [11] we defined a contractive IFS and, it was shown there that in a T_1 compact space X, the Hutchinson operator generated by a contractive IFS, if it is closed, is a topological contraction on the hyperspace of all nonempty closed subsets of X. Applying the analogue of the Banach fixed point theorem to the hyperspace and such an IFS and its Hutchinson operator, we conclude that it has a unique fixed point. We show here that in a T_1 compact topological space, the Hutchinson operator of a contractive IFS may not be closed in the hyperspace, but, nevertheless, the Hutchinson operator of a contractive IFS always has a unique fixed point.

The paper is organized as follows: in the second section we recall the basic notations and terminology. In the third section we state and prove an analogue of the Banach fixed point theorem for T_1 Čech complete spaces. In the fourth section we prove two analogues of the Banach fixed point theorem for Hausdorff spaces. In the fifth section, as an application of the results in the previous section, we state and prove fixed point theorems for compact subsets of linear spaces with weak topologies and for compact monoids. In the next two sections we consider T_1 locally Hausdorff spaces and T_1 peripherally Hausdorff spaces and prove fixed point theorems for these classes of spaces. The final eighth section is devoted to the Hutchinson operator in T_1 compact spaces.

2. Notations and terminology

All topological spaces considered in this article are assumed to be nonempty. Let X be a topological space and $A \subseteq X$. A point $x \in X$ is called an *accumulation point* of A if in every open set containing x there exists a point from A different to x. A point x is called a *cluster point* of a sequence $(x_k)_k$ if for every open set U containing x, $x_k \in U$ for infinitely many ks. A Tychonoff topological space is Čech complete if it is a G_δ subset of any of its compactifications. An equivalent definition says that a Tychonoff topological space is Čech complete if there exists a sequence $(\mathcal{U}_i)_i$ of its open covers such that if a sequence $(F_m)_m$ of closed sets is centered and for each i there exists m_i such that F_{m_i} is a subset of some member of $\mathcal{U}_i$, then the intersection $\bigcap_m F_m$ is nonempty [5]. The condition that the space is Tychonoff in the second version of the definition is added to retain this condition from the first definition. However, the second definition makes sense if we omit this assumption. Further, we call a topological space *Čech complete* if it satisfies the second definition without assuming that it is Tychonoff.

A metric space (X, d) is an *Atsuji space* if for each open cover $\mathcal{U}$ of X there exists $\varepsilon > 0$ such that for every $x \in X$ the open ball $B_d(x, \varepsilon)$ (of radius ε and center x) is contained in some element U of the cover (comp. [1]). It is a classical fact, called the Lebesgue number lemma, that each compact metric space is an Atsuji space [12].

The class of all ordinal numbers will be denoted by On.

As usual, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ denotes the set of all integers.

A topological space X is *locally Hausdorff* if every point of the space has an open neighborhood U such that the topology of X restricted to U is Hausdorff.

A topological space is a 0-*Hausdorff space* if it is a singleton $\{x\}$. Let us assume that $0 < \alpha \in \mathrm{On}$ and β-Hausdorff spaces have been defined for all $\beta < \alpha$. A T_1 topological space is an *α-Hausdorff space* if for every point x in this space the subspace consisting of points that cannot be separated from x in a Hausdorff way (i.e., by two disjoint neighborhoods)

$$[x] := \bigcap \{\mathrm{cl}(U) : U \text{ is an open neigborhood of } x\},$$

is a β-Hausdorff space for some $\beta < \alpha$ depending on x.

Note that "1-Hausdorff" means simply "Hausdorff".

If X is an α-Hausdorff space for some ordinal α, then X will be called a *peripherally Hausdorff* space. It is easy to notice that the classes of α-Hausdorff spaces form a non-decreasing trans-finite sequence with respect to ordinals α.

Let X be a peripherally Hausdorff space. We define a *Hausdorff rank* of X:

$$\mathrm{rank}_{T_2}(X) = \min\{\alpha : X \text{ is an } \alpha\text{-Hausdorff space}\}.$$

In Section 7 we present some basic properties of α-Hausdorff spaces.

Let X be a fixed set. Let $f : X \to X$. Then f^n denotes the composition of n copies of f: $f^n := f \circ f \circ \ldots \circ f$; f^0 denotes the identity mapping on X.

Let X be a topological space. A mapping $f : X \to X$ is called *a topological contraction* if f is a closed mapping and for every open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ there exist $n \in \mathbb{N}$ and $\alpha \in \Lambda$ such that $f^n[X] \subseteq U_\alpha$ [11]. We do not require here that f be continuous. If X is a compact T_1 space, then (Theorem 4 in [11]), $f : X \to X$ is a topological contraction if and only if f is closed and

($\star$) for each two different points $x, y \in X$ there exists $n \in \mathbb{N}$ such that $f^n[X] \subseteq X \setminus \{x\}$ or $f^n[X] \subseteq X \setminus \{y\}$.

If X is a topological space, then a mapping $f : X \to X$ will be called *a feeble topological contraction* if for every open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X and every pair of points $x, y \in X$ there exist $n \in \mathbb{N}_0$ and $\alpha \in \Lambda$ such that $f^n[\{x, y\}] \subseteq U_\alpha$ [11]. In this definition we do not require continuity or closedness of f. We do so because we will consider both versions, continuous and closed, of feeble topological contractions.

If X is a topological space, then a mapping $f : X \to X$ will be called *a feeble+ topological contraction* if for every open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X and every pair of points x, y there exist $n_0 \in \mathbb{N}_0$ and $\alpha \in \Lambda$ such that $f^n[\{x, y\}] \subseteq U_\alpha$ for all $n \geq n_0$. This definition also does not require continuity or closedness of f.

For a topological space X, an *iterated function system* (for short: IFS) is any finite family $\mathcal{F} = \{f_1, \ldots, f_m\}$ of closed mappings from X to X. We say that an IFS $\mathcal{F} = \{f_1, \ldots, f_m\}$ is *contractive* if for any open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of X there exists $n \in \mathbb{N}$ such that for any sequence $(i_1, \ldots, i_n)$, $1 \leq i_j \leq m$ there exists U_α from the cover $\mathcal{U}$ such that the set $f_{i_1} \circ \ldots \circ f_{i_n}[X]$ is contained in U_α (in [2] such an IFS is called *globally contracting*). Thus $f : X \to X$ is a topological contraction if the one-element IFS consisting of the mapping f is contractive.

Let X be a T_1 topological space. Let 2^X be the family of all closed nonempty subsets of X. The Vietoris topology on 2^X is generated by the basis consisting of all sets of the form

$$S(V_0; V_1, \ldots, V_k) := \{K \in 2^X : K \subseteq V_0, K \cap V_i \neq \emptyset, 1 \leq i \leq k\},$$

where $k \in \mathbb{N}$ and $V_0, V_1, \ldots, V_k$ are open subsets of X. The space 2^X with the Vietoris topology is called the *hyperspace* of X. It is known that if X is a T_1 compact space, then so is its hyperspace 2^X [10].