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ON THE SIMILARITY BETWEEN THE IWASAWA
 PROJECTION AND THE DIAGONAL PART

by

J.J. Duistermaat

1. Statement of the result.

Let G be a real connected semisimple Lie group with finite center and $G = KAN$ its Iwasawa decomposition. Via the adjoint representation, and with respect to a suitable basis in $\mathfrak{g}$, K , resp. A , resp. N are the set of matrices in G which are orthogonal, resp. diagonal with positive entries, resp. upper triangular.

The Iwasawa projection H from G onto the Lie algebra $\mathfrak{a}$ of A is defined by

$$(1.1) \quad x \in K \cdot \exp H(x) \cdot N, \quad x \in G.$$

Obviously H factorizes through the projection from G onto the (non-compact Riemannian) symmetric space $K \backslash G$. If $\mathfrak{s}$ (called $\mathfrak{p}$ by everybody else) denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the killing form, then the Cartan decomposition $G = K \cdot \exp \mathfrak{s}$ yields that

$$(1.2) \quad \mathfrak{s} \xrightarrow{\exp} G \rightarrow K \backslash G$$

is a diffeomorphism from $\mathfrak{s}$ onto $K \backslash G$. So the Iwasawa projection can be studied by looking at the mapping

$$(1.3) \quad \gamma = H \circ \exp : \mathfrak{s} \rightarrow \mathfrak{a}.$$

On the other hand we have the orthogonal projection

$$(1.4) \quad \pi : \mathfrak{s} \rightarrow \mathfrak{a}$$

with respect to the Killing form. In the above matrix terminology, $\mathfrak{s}$ is the space of symmetric matrices in $\mathfrak{g}$ and π is the operation of taking the diagonal part of the symmetric matrix. So this projection has a very simple minded interpretation, whereas the Iwasawa projection is a rather more mysterious object.

Theorem 1.1. There is a real analytic map $\Psi : \mathfrak{s} \rightarrow K$ such that

- i) $\Phi_X : k \rightarrow k \cdot \Psi(\text{Ad } k^{-1}(X))$ is a diffeomorphism from K onto K , for each $X \in \mathfrak{s}$.
- ii) $\gamma(\text{Ad } \Psi(X)^{-1}(X)) = \pi(X)$ for all $X \in \mathfrak{s}$.

That is, we can turn the Iwasawa projection into the orthogonal projection by an action of $\text{Ad } K$, the element of K depending analytically on $X \in \mathfrak{s}$.

It also follows from the theorem that the images of an $\text{Ad } K$ -orbit in $\mathfrak{s}$ under γ and π are the same. This was obtained before by Kostant [4] who showed separately that both images are equal to the convex hull of the intersection of the $\text{Ad } K$ -orbit in $\mathfrak{s}$ with $\mathfrak{a}$. Since this intersection is equal to a Weyl group orbit in $\mathfrak{a}$, which is finite, this image is a convex polytope. Very remarkable because an $\text{Ad } K$ -orbit is such a roundish object!

Later Heckman [3] reduced the convexity theorem for the Iwasawa projection to the convexity theorem for the diagonal part, for which the proof is much simpler, using a homotopy argument. This homotopy argument actually is one of the elements in the proof of Theorem 1.1.

For me the major motivation for wanting the theorem was the study in [2], together with Kolk and Varadarajan, of the asymptotic behaviour of integrals of the form

$$(1.5) \quad I_a(X, \xi) = \int_K e^{i\langle \gamma(\text{Ad } k^{-1}(X)), \xi \rangle} a(X, k) dk$$

as $\|\xi\| \rightarrow \infty$, $\xi \in \mathfrak{a}^*$. The matrix coefficients of the principal series representations of G are given by such integrals, the simplest case being the elementary spherical functions where

$$(1.6) \quad a(X, k) = e^{-\langle \gamma(\text{Ad } k^{-1}(X)), \rho \rangle}.$$

The idea in [2] was to consider (1.5) as an oscillatory integral, for which the asymptotics is concentrated at the stationary points of the "phase function".

$$(1.7) \quad F_{X, \xi} : k \rightarrow \langle \gamma(\text{Ad } k^{-1}(X)), \xi \rangle$$

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on K . We then observed that $F_{X,\xi}$ had exactly the same critical points and critical values as its "infinitesimal counterpart"

$$(1.8) \quad f_{X,\xi} = \lim_{t \rightarrow 0} \frac{1}{t} F_{tX,\xi} : k \rightarrow \pi(\text{Ad } k^{-1}(X)), \xi >.$$

These critical points in turn had such a special, rigid structure that the asymptotics of (1.5) could be obtained by a repeated application of the method of stationary phase.

It had already been observed in [2] that the equality of critical points and critical values of $F_{X,\xi}$ and $f_{X,\xi}$ leads to the existence of a diffeomorphism $\phi_{X,\xi} : K \rightarrow K$ such that $F_{X,\xi} \circ \phi_{X,\xi} = f_{X,\xi}$.

However, the diffeomorphism is not unique and at that time I could not find $\phi_{X,\xi}$ depending smoothly on X and ξ . Already continuous dependence on ξ would imply, replacing ξ by $t\xi$, dividing by t , and letting $t \rightarrow 0$, that $F_{X,\xi} \circ \phi_{X,0} = f_{X,\xi}$. That is, one could find a diffeomorphism ϕ_X not depending on ξ . Then, using the substitution of variables

$$(1.9) \quad k = \phi_X(l), \quad l \in K,$$

the integral (1.5) can be rewritten as ($X \in \mathfrak{s}$, $\xi \in \mathfrak{a}^*$)

$$(1.10) \quad I_a(X, \xi) = \int_K e^{i\langle \pi(\text{Ad } k^{-1}(X)), \xi \rangle} a(X, \phi_X(k)) \cdot \left| \det \frac{\partial \phi_X}{\partial k}(k) \right| dk.$$

In this way the study of the asymptotic behaviour would be reduced to doing stationary phase with the simpler $f_{X,\xi}$ as the phase function, rather than $F_{X,\xi}$. (Such asymptotics has been done before by Clerc and Barlet [1].)

It is one of the applications of Theorem 1.1, that the integral representation (1.10) actually holds with a $\phi_X(k)$ which depends analytically on X and k simultaneously. For instance, for the elementary spherical functions this leads to an integral formula of the form

$$(1.11) \quad \phi_\xi(\exp X) = \int_K e^{i\langle \pi(\text{Ad } k^{-1}(X)), \xi \rangle} b(\text{Ad } k^{-1}(X)) dk,$$

for some analytic function $b : \mathfrak{s} \rightarrow \mathbb{R}$. As an application of the analyticity of b , one can note that replacing ξ , resp. X by $i\xi$, resp. iX , one obtains the elementary spherical functions for the compact symmetric space which is dual to $K \backslash G$. (In this case ξ has to be taken in a weight lattice.) So also for these functions an integral formula like (1.10) holds, at least for small $\|X\|$. I owe

this observation to Richard van den Dries (T.H. Delft), who is using this integral formula in his characterization of invariant pseudo-differential operators on compact symmetric spaces in terms of their eigenvalues.

2. $SL(2, \mathbb{R})$.

For $G = SL(2, \mathbb{R})$, $\dim K = \dim \mathfrak{a}$ ($= 1$), so the substitution of variables is unique up to a flip. In order to determine it explicitly, write the elements of K as

$$(2.1) \quad k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

and the elements of $\mathfrak{a}$ as

$$(2.2) \quad X = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Then $Y = \text{Ad } k^{-1}(X) = k^{-1}Xk$ is the general element of $\mathfrak{s}$, and $\phi_X(k)$ is the element of K with the coordinate μ given implicitly by

$$(2.3) \quad e^{2t} \cos^2 \mu + e^{-2t} \sin^2 \mu = e^{2t \cos 2\theta}.$$

From this one can determine $\Psi(Y) = k^{-1} \cdot \phi_X(k)$. It is not entirely trivial to verify that this defines a real analytic mapping $\Psi : \mathfrak{s} \rightarrow \mathfrak{a}$.

The Jacobian of ϕ_X is equal to

$$(2.4) \quad \frac{2|t \sin 2\theta| \cdot e^{t \cos 2\theta}}{\sqrt{2}(\cosh(2t) - \cosh(2t \cos 2\theta))},$$

leading to the following formula for the elementary spherical function:

$$(2.5) \quad \phi_\xi(\exp X) = \frac{1}{2\pi} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{it\tau \cos 2\theta} \frac{|t \sin 2\theta|}{\sqrt{\frac{\cosh(2t) - \cosh(2t \cos 2\theta)}{2}}} d\theta.$$

Here we have written $\langle X, \xi \rangle = t\tau$. This can also be written as

$$(2.6) \quad \phi_\xi(\exp X) = \frac{4}{\pi} \int_0^t \cos \tau s \cdot \frac{ds}{\sqrt{4(\cosh(2t) - \cosh(2s))}}.$$

A similar formula for all rank one symmetric spaces can be found in Koornwinder [8], formula (2.16) and (2.18).

I prefer (2.5) over (2.6), because there are no boundary points nor singularities for the integrand as in (2.6). To see the analyticity of the integrand in (2.5) we write