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DECOMPOSITION MATRICES FOR CATEGORY $\mathcal{O}$ OF RATIONAL CHEREDNIK ALGEBRAS

BY EMILY NORTON

ABSTRACT. — We find the decomposition matrices of the Category $\mathcal{O}$ for rational Cherednik algebras of Coxeter groups of types H_4, F_4, E_6, E_7 and E_8. The decomposition matrix is an important numerical invariant of the Category $\mathcal{O}$: the matrix entries are the multiplicities of irreducible modules in standard modules (also called Verma modules). In the case of F_4, E_6, E_7 and E_8, the parameter is not a half-integer and for E_8 we only consider the blocks of defect 2. In the case of F_4, we consider the case of equal parameters only. As a consequence, we obtain character formulas for irreducible representations in the Category $\mathcal{O}$, as well as a classification of the finite-dimensional representations of these rational Cherednik algebras.

RÉSUMÉ (*Matrices de décomposition pour la catégorie $\mathcal{O}$ des algèbres de Cherednik rationnelles*). — Nous trouvons les matrices de décomposition de la catégorie $\mathcal{O}$ pour les algèbres de Cherednik rationnelles des groupes de Coxeter de types H_4, F_4, E_6, E_7 et E_8. La matrice de décomposition est un invariant numérique important de la catégorie $\mathcal{O}$: les entrées de la matrice sont les multiplicités des modules irréductibles dans les modules standard, dit Verma. Dans le cas de F_4, E_6, E_7 et E_8, le paramètre n'est pas un demi-entier et dans le cas de E_8, nous ne considérons que les blocs de défaut 2. Dans le cas de F_4, nous ne considérons que le cas des paramètres égaux. En conséquence, nous obtenons des formules de caractères pour les représentations irréductibles dans la catégorie $\mathcal{O}$, ainsi qu'une classification des représentations de dimension finie de ces algèbres de Cherednik rationnelles.

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Introduction

Rational Cherednik algebras $H_c(W)$ for any complex reflection group W (and depending on a parameter c) were defined by Etingof and Ginzburg [10]. We are interested in fundamental questions about their representation theory such as

- which irreducible representations are finite-dimensional and what are their dimensions?
- what are the graded characters of the irreducible representations?
- what are the multiplicities of irreducible representations in standard (i.e., Verma) modules?

These questions have been answered when W is a dihedral group by Chmutova [6], when W is the Coxeter group H_3 by Balagovic–Puranik [1], and when W is the wreath product of a cyclic group with a symmetric group by several groups of authors [28, 34, 22]. Calculating the decomposition matrix of the category $\mathcal{O}_c(W)$ associated to $H_c(W)$ yields the answers to all three questions. The category $\mathcal{O}_c(W)$, studied in [15], is a lowest-weight category defined analogously to the category $\mathcal{O}$ of a complex semisimple Lie algebra. It contains Verma modules $\mathrm{M}(\tau)$ and all finite-dimensional representations of $H_c(W)$. The decomposition matrix of $\mathcal{O}_c(W)$ encodes the multiplicities $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ of irreducible representations $\mathrm{L}(\sigma)$ appearing in the composition series of Verma modules $\mathrm{M}(\tau)$. The decomposition matrix is an important invariant of the category $\mathcal{O}_c(W)$.

This paper seeks to answer the three questions above for W being one of the following exceptional Coxeter groups: H_4, F_4 (in the case of equal parameters), E_6, E_7, and E_8. The main result of this paper is the computation of the decomposition matrices of the categories $\mathcal{O}_c(W)$ for all parameters $c = 1/d$ with d a divisor of one of the fundamental degrees of W and $d > 2$ (and in the case of E_8, if $d \in \{3, 4, 6\}$, then we also exclude the decomposition matrix of the principal block).

The character of $\mathrm{L}(\tau)$ may be read off the row labeled by τ in the inverse of the decomposition matrix, from which the graded character of $\mathrm{L}(\tau)$ follows from a simple formula. The graded character is a rational function in t, and the order of its pole at $t = 1$ coincides with the dimension of the support of $\mathrm{L}(\tau)$. The irreducible representations with support of dimension less than the rank of W are of particular interest; these are the representations killed by the Knizhnik–Zalomodchikov functor. Among them, the representations whose support has dimension 0 consist of the finite-dimensional representations. Evaluating the character of a finite-dimensional representation at $t = 1$ gives its dimension. Both the characters of finite-dimensional representations, which are polynomials in $\mathbb{Z}[t^{-1}, t]$ symmetric under interchanging t and t^{-1}, and the dimensions of finite-dimensional representations have meanings in other contexts. The characters should be Markov traces of braids of type W, as has been studied in [16] and [32]. In some cases, the dimensions of finite-dimensional representations are dimensions of the homology of affine Springer fibers [25], [33]. And, according to the philosophy of Miyachi, the finite-dimensional representations of the rational Cherednik algebra should correspond to cuspidal modular representations of a finite group of Lie type with Weyl group W.

The decomposition matrices we compute in this paper for $\mathcal{O}_c(W)$ turn out to be closely related to the unipotent decomposition matrices of exceptional finite groups of Lie type in cross-characteristic. If $G(q)$ is a finite group of Lie type with Weyl group W, then its representation theory in characteristic ℓ when ℓ divides the order of $G(q)$ and $\ell \nmid q$ depends on which cyclotomic polynomial $\Phi_d(q)$ the prime ℓ divides, provided that ℓ is large enough. The relevant integers d are the divisors of the fundamental degrees of W. In many of the examples in this paper, the decomposition matrix of $\mathcal{O}_c(W)$ embeds as a square submatrix in the unipotent decomposition matrix of $G(q)$, for $\ell >> 0$ dividing $\Phi_d(q)$ with $c = 1/d$. In particular, this is true for all the blocks of $\mathcal{O}_c(W)$ for $W = E_6$, E_7, and E_8 at $c = 1/4$ corresponding to the unipotent decomposition matrices at $d = 4$ computed by Dudas and Malle [8].

Let d divide one of the fundamental degrees of W and let $\ell | \Phi_d(q)$ be a prime. Let $\mathcal{B}$ be a block of $\mathcal{O}_c(W)$ and take any $\tau \in \operatorname{Irr} W$ such that $\mathrm{L}(\tau) \in \mathcal{B}$. Let $G(q)$ be a finite group of Lie type with Weyl group W and let $\widetilde{\mathcal{B}}$ be the unipotent block of $G(q)$ over a field of characteristic $\ell >> 0$ containing the unipotent representation labeled by $\tau \otimes \mathrm{sgn}$. Let $\sigma \in \operatorname{Irr} W$. The following conjecture is naively based on the existing low-rank examples:

Conjecture 1. — *The rational Cherednik algebra decomposition number* $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ *is a lower bound for the $G(q)$ decomposition number in row* $\tau \otimes \mathrm{sgn}$ *and column* $\sigma \otimes \mathrm{sgn}$ *of the decomposition matrix of the unipotent block* $\widetilde{\mathcal{B}}$.

Note that the relevant square submatrix of the decomposition matrix of $G(q)$ is the one labeled by unipotent characters lying in the principal series. Also, note that the conjecture holds for those columns corresponding to simple representations of the Hecke algebra at a d'th root of unity, since this rectangular matrix embeds in both the rational Cherednik algebra decomposition matrix by [5], [4], and the $G(q)$ decomposition matrix for $\ell >> 0$ by James' conjecture. One might imagine that what happens exactly is that the columns of the $G(q)$ decomposition matrix are nonnegative linear combinations of the columns of the $\mathcal{O}_c(W)$ decomposition matrix.

We have only stated the conjecture for the principal series, but a version of the conjecture should also hold for the other Harish–Chandra series of $G(q)$, which are parameterized by cuspidal representations in characteristic 0.

1. Rational Cherednik algebras and their character theory

Rational Cherednik algebras were defined by Etingof and Ginzburg in [10]. They can be constructed starting from any complex reflection group. In this paper, we will only treat rational Cherednik algebras associated to finite Coxeter groups. In particular, many of our techniques are valid only in this setting. The representation theory of rational Cherednik algebras for exceptional Coxeter groups was provided by Chmutova for dihedral groups [6] and Balagovic–Puranik for H_3 [1]. In this section, we recall the background material on rational Cherednik algebras $H_c(W)$ for finite Coxeter groups W and their representations in the category $\mathcal{O}_c(W)$ that we need to compute the decomposition matrix of $\mathcal{O}_c(W)$. Only Section 1.8 contains original material.

1.1. Rational Cherednik algebras. — Let W be a finite Coxeter group. Let $\mathfrak{h}$ denote the reflection representation of W over $\mathbb{C}$ and $\mathfrak{h}^* \cong \mathfrak{h}$ its dual representation. Let $\operatorname{rk} W$ denote the dimension of $\mathfrak{h}$ as a $\mathbb{C}$-vector space, which is equal to the number of simple reflections generating W. Denote by $S \subset W$ the set of *all* reflections in W. We will denote by $\operatorname{Irr} W$ a complete set of isomorphism classes of irreducible representations of W over $\mathbb{C}$. Given $\tau \in \operatorname{Irr} W$, we will write $\tau' := \tau \otimes \operatorname{sign} \cong \tau \otimes \bigwedge^{\operatorname{rk} W} \mathfrak{h}^*$.

To W we will associate an associative $\mathbb{C}$-algebra $H_c(W)$ called the rational Cherednik algebra, depending on a parameter $c \in \mathbb{C}$. More generally, we may define $H_c(W)$ depending on a collection c of parameters, one for each W-conjugacy class of reflections. Thus $H_c(W)$ depends on a single parameter if W is of types A, D, E, or H or if it is a dihedral group $I_2(m)$ with m odd, and it depends on two parameters if W is of types B, F, or G or if it is a dihedral group $I_2(m)$ with m even. When W has two conjugacy classes of reflections, the case that the two parameters are equal is a basic and important one, as it arises naturally from geometry [25],[33]. We will exclusively study the case of equal parameters in this paper.

Fix a parameter $c \in \mathbb{C}$. Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra on $\mathfrak{h} \oplus \mathfrak{h}^*$ and let $\mathbb{C}[W]$ be the group algebra of W over $\mathbb{C}$. Since W acts on $\mathfrak{h}$ and $\mathfrak{h}^*$, we may form the semidirect product algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[W]$.

Definition 1.1. — [10] The rational Cherednik algebra $H_c(W)$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[W]$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = (y, x) - c \sum_{s \in S} (\alpha_s^\vee, x)(y, \alpha) s$$

for all $x, x' \in \mathfrak{h}^*$ and all $y, y' \in \mathfrak{h}$. Here, $(-,-)$ is the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, and $\alpha_s \in \mathfrak{h}^*$, $\alpha_s^\vee \in \mathfrak{h}$ are dual eigenvectors for the reflection s, normalized so that $(\alpha_s, \alpha_s^\vee) = 2$.

An important first result about $H_c(W)$, known as the PBW theorem by analogy with the triangular decomposition of the universal enveloping algebra of a complex semisimple Lie algebra, states that elements of $H_c(W)$ may be put in normal form with respect to a preferred order of x's, y's, and elements of $\mathbb{C}[W]$.

THEOREM 1.2. — *(Etingof–Ginzburg,* [10, Theorem 1.3]*) There is an isomorphism of* $\mathbb{C}$*-vector spaces* $H_c(W) \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}]$.

There are subalgebras $\mathbb{C}[\mathfrak{h}] \rtimes \mathbb{C}[W] \subset H_c(W)$ and $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W] \subset H_c(W)$, analogous to positive and negative Borel subalgebras. We remark that fixing dual bases $y_1, \dots, y_n$ and $x_1, \dots, x_n$ for $\mathfrak{h}$ and $\mathfrak{h}^*$, $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h}) \cong \mathbb{C}[y_1, \dots, y_n]$ denotes polynomials in the y-variables, and $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*) \cong \mathbb{C}[x_1, \dots, x_n]$ denotes polynomials in the x-variables.

1.2. Category $\mathcal{O}$ for rational Cherednik algebras. — We now discuss some of the representation theory of $H_c(W)$. Throughout the text, we will use the terms "simple module" and "irreducible representation" to mean the same thing. The main types of modules or representations that we will be interested in are simple modules, standard (or Verma) modules, and projective modules. The standard modules have a concrete construction and description. We will study a certain nice category of $H_c(W)$-modules containing the standard modules and their simple composition factors. All of these modules are infinite-dimensional except for some simple modules. The finite-dimensional simple modules are rare and play an important role in the theory.

The category $\mathcal{O}_c(W)$ is defined to be the full subcategory of left $H_c(W)$-modules whose objects consist of those representations of $H_c(W)$ that are (i) finitely generated over $\mathbb{C}[\mathfrak{h}]$ and (ii) locally nilpotent for $\mathbb{C}[\mathfrak{h}^*]$ [15, Section 2.2, Section 3.2.1]. This definition follows the definition of category $\mathcal{O}$ of a complex semisimple Lie algebra using the triangular decomposition given by the PBW theorem. By [15, Theorem 2.19], the category $\mathcal{O}_c(W)$ is a highest-weight category in the sense of [7]. However, we will use lowest weights instead of highest weights. The structure of the category $\mathcal{O}_c(W)$ depends on the parameter c. For generic $c \in \mathbb{C}$, $\mathcal{O}_c(W)$ is semisimple and equivalent to $\mathbb{C}[W]$-mod. We will assume from now on that $c \in \mathbb{R}$, as this is a precondition for $\mathcal{O}_c(W)$ to not be semisimple. The relevant parameters are discussed in Section 1.6.

Given $\tau \in \mathrm{Irr}\, W$, we define the *standard module with lowest weight* τ as follows:

$$\mathrm{M}(\tau) := H_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]} \tau.$$

In this construction, τ is considered as a representation of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ by letting W act as it acts on τ and extending this to an action of $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$ by having $\mathfrak{h}$ act by 0. We then induce the resulting representation from $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$ to $H_c(W)$ to obtain $\mathrm{M}(\tau)$. The standard modules $\mathrm{M}(\tau)$, $\tau \in \mathrm{Irr}\, W$, are the