

370

ASTÉRISQUE

2015

DE LA GÉOMÉTRIE ALGÈBRIQUE
AUX FORMES AUTOMORPHES (II)

J.-B. BOST, P. BOYER, A. GENESTIER,
L. LAFFORGUE, S. LYSENKO, S. MOREL, B.C. NGÔ, eds.

OUTLINE OF THE PROOF OF THE
GEOMETRIC LANGLANDS CONJECTURE FOR GL_2

Dennis GAITSGORY

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 370, 2015

Comité de rédaction

Ahmed ABBES	Damien GABORIAU
Viviane BALADI	Michael HARRIS
Gérard BESSON	Fabrice PLANCHON
Laurent BERGER	Pierre SCHAPIRA
Philippe BIANE	Bertrand TOËN
Hélène ESNAULT	
Éric VASSEROT (dir.)	

Diffusion

Maison de la SMF	Hindustan Book Agency	AMS
Case 916 - Luminy	O-131, The Shopping Mall	P.O. Box 6248
13288 Marseille Cedex 9	Arjun Marg, DLF Phase 1	Providence RI 02940
France	Gurgaon 122002, Haryana	USA
smf@smf.univ-mrs.fr	Inde	www.ams.org

Tarifs

Vente au numéro : 98 € (\$ 147)

Abonnement Europe : 650 €, hors Europe : 689 € (\$ 1033)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Astérisque

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie

75231 Paris Cedex 05, France

Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96

revues@smf.ens.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2015

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN 0303-1179

ISBN 978-2-85629-806-0

Directeur de la publication : Marc Peigné

OUTLINE OF THE PROOF OF THE GEOMETRIC LANGLANDS CONJECTURE FOR GL_2

by

Dennis Gaitsgory

To Gérard Laumon

Abstract. — We outline a proof of the categorical geometric Langlands conjecture for GL_2 , as formulated in [AG], modulo a number of more tractable statements that we call Quasi-Theorems.

Résumé (Les grandes lignes de la démonstration de la correspondance de Langlands géométrique pour GL_2). — On donne les grandes lignes d’une démonstration de la conjecture de Langlands géométrique catégorique pour GL_2 , telle que formulée dans [AG], modulo un certain nombre d’énoncés davantage à portée que nous appelons des quasi-théorèmes.

Introduction

0.1. The goal of this paper. — The goal of this paper is to describe work-in-progress by D. Arinkin, V. Drinfeld and the author⁽¹⁾ towards the proof of the (categorical) geometric Langlands conjecture.

The contents of the paper can be summarized as follows: we reduce the geometric Langlands conjecture to a combination of two sets of statements.

The first set is what we call “quasi-theorems”. These are plausible (and tractable) statements that involve Langlands duality, but either for proper Levi subgroups, or of local nature, or both. Hopefully, these quasi-theorems will soon turn into actual theorems.

The second set are two conjectures (namely, Conjectures 8.2.9 and 10.2.8), both of which are theorems for GL_n . However, these conjectures *do not* involve Langlands

2010 Mathematics Subject Classification. — 14F05, 14H60.

Key words and phrases. — Geometric Langlands correspondence, derived algebraic geometry.

1. The responsibility for any deficiency or undesired outcome of this paper lies with the author of this paper.

duality: Conjecture 8.2.9 only involves the geometric side of the correspondence, and Conjecture 10.2.8 only the spectral side.

0.2. Strategy of the proof. — In this subsection we will outline the general scheme of the argument. We will be working over an algebraically closed field k of characteristic 0. Let X be a smooth and complete curve over k , and G a reductive group. We let $\check{G}$ denote the Langlands dual group, also viewed as an algebraic group over k .

0.2.1. Formulation of the conjecture. — The categorical geometric Langlands conjecture is supposed to compare two triangulated (or rather DG categories). One is the “geometric” (or “automorphic”) side that has to do with D-modules on the stack Bun_G of G -bundles on X . The other is the “spectral” (or “Galois”) side that has to do with quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\check{G}}$ on $\check{G}$ -local systems on X .

In our formulation of the conjecture, the geometric side is taken “as is”. *I.e.*, we consider the DG category $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ of D-modules on Bun_G . We refer the reader to [DrGa2] for the definition of this category and a discussion of its general properties (*e.g.*, this category is *compactly generated* for non-tautological reasons).

A naive guess for the spectral side is the DG category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$. However, this guess turns out to be slightly wrong, whenever G is not a torus. A quick way to see that it is wrong is *via* the compatibility of the conjectural geometric Langlands equivalence with the functor of Eisenstein series, see Property Ei stated in Sect. 6.4.8. Namely, if P is a parabolic of G with Levi quotient M , we have the Eisenstein series functors

$$\begin{aligned} \mathrm{Eis}_P : \mathrm{D}\text{-mod}(\mathrm{Bun}_M) &\longrightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \quad \text{and} \\ \mathrm{Eis}_{\check{P}, \mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{M}}) &\longrightarrow \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}), \end{aligned}$$

that are supposed to match up under the geometric Langlands equivalence (up to a twist by some line bundles). However, this cannot be the case because the functor Eis_P preserves compactness (see [DrGa3]), whereas $\mathrm{Eis}_{\check{P}, \mathrm{spec}}$ does not.

Our “fix” for the spectral side is designed to make the above problem with Eisenstein series go away in a minimal way (see Proposition 6.4.7). We observe that the non-preservation of compactness by the functor $\mathrm{Eis}_{\check{P}, \mathrm{spec}}$ has to do with the fact that the stack $\mathrm{LocSys}_{\check{G}}$ is not smooth. Namely, it expresses itself in that some coherent complexes on $\mathrm{LocSys}_{\check{G}}$ are non-perfect.

Our modified version for the spectral side is the category that we denote

$$\mathrm{IndCoh}_{\mathrm{Nilp}_G^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}),$$

see Sect. 3.3.2. It is a certain enlargement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, whose definition uses the fact that $\mathrm{LocSys}_{\check{G}}$ is a derived locally complete intersection, and the theory of singular support of coherent sheaves for such stacks developed in [AG].

0.2.2. *Idea of the proof.* — The idea of the comparison between the categories $\mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ and $\mathrm{IndCoh}_{\mathrm{Nilp}_G^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ pursued in this paper is the following: we embed each side into a more tractable category and compare the essential images.

For the geometric side, the more tractable category in question is the category that we denote $\mathrm{Whit}^{\mathrm{ext}}(G, G)$, and refer to it as the *extended Whittaker category*; the nature of this category is explained in Sect. 0.2.3 below. The functor

$$\mathrm{D}\text{-mod}(\mathrm{Bun}_G) \longrightarrow \mathrm{Whit}^{\mathrm{ext}}(G, G)$$

(which, according to Conjecture 8.2.9, is supposed to be fully faithful) is that of *extended Whittaker coefficient*, denoted $\mathrm{coeff}_{G,G}^{\mathrm{ext}}$.

For the spectral side, the more tractable category is denoted $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$, and the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_G^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow \mathrm{Glue}(\check{G})_{\mathrm{spec}}$$

is denoted by $\mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}})$ (this functor is fully faithful by Theorem 9.3.8). The idea of the pair $(\mathrm{Glue}(\check{G})_{\mathrm{spec}}, \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}))$ is explained in Sect. 0.2.4.

We then claim (see Quasi-Theorems 9.4.2 and 9.4.5) that there exists a canonically defined fully faithful functor

$$\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \longrightarrow \mathrm{Whit}^{\mathrm{ext}}(G, G).$$

Thus, we have the following diagram

$$(0.1) \quad \begin{array}{ccc} \mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}}} & \mathrm{Whit}^{\mathrm{ext}}(G, G) \\ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \uparrow & & \uparrow \mathrm{coeff}_{G,G}^{\mathrm{ext}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_G^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) & & \mathrm{D}\text{-mod}(\mathrm{Bun}_G), \end{array}$$

with all the arrows being fully faithful.

Assume that the essential images of the functors

$$(0.2) \quad \mathbb{L}_{G,G}^{\mathrm{Whit}^{\mathrm{ext}}} \circ \mathrm{Glue}(\mathrm{CT}_{\mathrm{spec}}^{\mathrm{enh}}) \text{ and } \mathrm{coeff}_{G,G}^{\mathrm{ext}}$$

coincide. We then obtain that diagram (0.1) can be (uniquely) completed to a commutative diagram by means of a functor

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_G^{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \longrightarrow \mathrm{D}\text{-mod}(\mathrm{Bun}_G),$$

and, moreover, $\mathbb{L}_G$ is automatically an equivalence.

The required fact about the essential images of the functors (0.2) follows from Conjecture 10.2.8.