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WEIGHT FILTRATION AND
SLOPE FILTRATION ON THE
RIGID COHOMOLOGY OF

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WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Yukiyoshi Nakkajima

Abstract. — We construct a theory of weights on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$ by using the log crystalline cohomology of a split proper hypercovering of the scheme. We also calculate the slope filtration on the rigid cohomology by using the cohomology of the log de Rham-Witt complex of the hypercovering.

Résumé. — Nous construisons une théorie des poids sur la cohomologie rigide d'un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$ en utilisant la cohomologie log-cristalline d'un hyperrecouvrement propre scindé du schéma. Nous calculons aussi la filtration par les pentes sur la cohomologie rigide en utilisant la cohomologie du complexe de de Rham-Witt logarithmique de l'hyperrecouvrement.

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1. Introduction

J.-P. Serre has conjectured the existence of the virtual Betti number of a separated scheme of finite type over a field (see [33]). Furthermore, A. Grothendieck has conjectured the existence of, so to speak, the virtual slope number and the virtual Hodge number of the scheme (see [loc. cit.]). Let us recall the numbers briefly as follows.

Let κ be an algebraically closed field of characteristic $p \geq 0$. Let $\mathbf{S}(\kappa)$ be the set of isomorphism classes of separated schemes of finite type over κ . For a separated scheme U of finite type over κ , we denote by $[U]$ the element of $\mathbf{S}(\kappa)$ defined by U . For a function $f: \mathbf{S}(\kappa) \rightarrow \mathbb{Z}$, we denote $f([U])$ by $f(U)$ for simplicity of notation. Let r be a nonnegative integer. Then Serre has conjectured that there exists a unique function

$$h_c^r: \mathbf{S}(\kappa) \rightarrow \mathbb{Z}$$

satisfying the following two properties (see [33]):

▷ For a closed subscheme Z of U ,

$$(I-a) \quad h_c^r(U) = h_c^r(Z) + h_c^r(U \setminus Z).$$

▷ If U is a proper smooth scheme over κ , then

$$(I-b) \quad h_c^r(U) = (-1)^r \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^r(U, \mathbb{Q}_\ell)$$

for a prime number $\ell \neq p$.

Here the letter ‘c’ in the notation h_c^r stands for the compact support. Heuristically $h_c^r(U)$ is given by the formula (see [loc. cit.]):

$$(1.0.1) \quad h_c^r(U) = \begin{cases} \sum_{m=0}^{\infty} (-1)^m \dim_{\mathbb{Q}_\ell} \text{gr}_r^P H_{\text{ét},c}^m(U, \mathbb{Q}_\ell) & (p > 0), \\ \sum_{m=0}^{\infty} (-1)^m \dim_{\mathbb{Q}} \text{gr}_r^P H_c^m(U_{\text{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}), \end{cases}$$

where P is the ‘weight filtration’ on $H_{\text{ét},c}^m(U, \mathbb{Q}_\ell)$ (resp. $H_c^m(U_{\text{an}}, \mathbb{Q})$). (For the cohomology without compact support, see (1.0.3) below for the definition of the weight filtration P when κ is $\overline{\mathbb{F}}_p$ (resp. $\mathbb{C}$).) Set

$$H_c^m(U) = \begin{cases} H_{\text{ét},c}^m(U, \mathbb{Q}_\ell) & (p > 0), \\ H_c^m(U_{\text{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}). \end{cases}$$

Heuristically (I-a) is obtained from the strict exactness with respect to P of the exact sequence

$$\cdots \rightarrow H_c^m(U \setminus Z) \rightarrow H_c^m(U) \rightarrow H_c^m(Z) \rightarrow \cdots$$

Serre has called $h_c^r(U)$ the virtual Betti number of $[U]$. He has also remarked that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h_c^r . Serre's conjecture was the starting point of Grothendieck's theory of weights (see [33]).

In [33] Grothendieck has conjectured, for two nonnegative integers i and j , there exists a function

$$h_c^{ij}: \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

such that

$$(II-a) \quad h_c^r(U) = \sum_{i+j=r} h_c^{ij}(U).$$

In the case $p > 0$, let $\mathcal{W}$ be the Witt ring of κ , K_0 the fraction field of $\mathcal{W}$ and $\mathcal{W}\Omega_U^i$ the de Rham-Witt sheaf of U of degree i (see [47]). We call $h_c^{ij}(U)$ the *virtual slope number* and the *virtual Hodge number* of $[U]$ if $p > 0$ and $p = 0$, respectively, because we require that h_c^{ij} satisfies the equation:

$$(II-b) \quad h_c^{ij}(U) = \begin{cases} (-1)^{i+j} \dim_{K_0}(H^j(U, \mathcal{W}\Omega_U^i) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ (-1)^{i+j} \dim_{\kappa} H^j(U, \Omega_{U/\kappa}^i) & (p = 0) \end{cases}$$

if U is a proper smooth scheme over κ . We also require that h_c^{ij} satisfies the equation

$$(II-c) \quad h_c^{ij}(U) = h_c^{ij}(Z) + h_c^{ij}(U \setminus Z).$$

Furthermore, we conjecture that h_c^{ij} is uniquely determined by the equations (II-b) and (II-c). It is easy to check that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h_c^{ij} . The existence of h_c^{ij} ($i, j \in \mathbb{N}$) implies the existence of h_c^r ($r \in \mathbb{N}$) because (I-a) is clear by (II-a) and (II-c) and because (I-b) holds as follows: if U is a proper smooth scheme over κ , then we have the equations

$$(1.0.2) \quad \dim_{\mathbb{Q}_\ell} H_{\text{ét}}^r(U, \mathbb{Q}_\ell) = \begin{cases} \dim_{K_0}(H_{\text{crys}}^r(U/\mathcal{W}) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \dim_{\kappa} H_{\text{dR}}^r(U/\kappa) & (p = 0), \end{cases}$$

$$= \begin{cases} \sum_{i+j=r} \dim_{K_0}(H^j(U, \mathcal{W}\Omega_U^i) \otimes_{\mathcal{W}} K_0) & (p > 0), \\ \sum_{i+j=r} \dim_{\kappa} H^j(U, \Omega_{U/\kappa}^i) & (p = 0). \end{cases}$$

The first (resp. the second) equation in (1.0.2) in the case $p > 0$ has been obtained in [54], [16] and [69] (resp. [47]). The first equation in (1.0.2) for