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WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

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WEIGHT FILTRATION AND SLOPE FILTRATION ON THE RIGID COHOMOLOGY OF A VARIETY IN CHARACTERISTIC $p > 0$

Yukiyoshi Nakkajima

Abstract. **—** We construct a theory of weights on the rigid cohomology of a separated scheme of finite type over a perfect field of characteristic $p > 0$ by using the log crystalline cohomology of a split proper hypercovering of the scheme. We also calculate the slope filtration on the rigid cohomology by using the cohomology of the log de Rham-Witt complex of the hypercovering.

Résumé. **—** Nous construisons une théorie des poids sur la cohomologie rigide d'un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$ en utilisant la cohomologie log-cristalline d'un hyperrecouvrement propre scindé du schéma. Nous calculons aussi la filtration par les pentes sur la cohomologie rigide en utilisant la cohomologie du complexe de de Rham-Witt logarithmique de l'hyperrecouvrement.

CONTENTS

1. Introduction

J.-P. Serre has conjectured the existence of the virtual Betti number of a separated scheme of finite type over a field (see [**33**]). Furthermore, A. Grothendieck has conjectured the existence of, so to speak, the virtual slope number and the virtual Hodge number of the scheme (see [*loc. cit.*]). Let us recall the numbers briefly as follows.

Let κ be an algebraically closed field of characteristic $p \geq 0$. Let $\mathbf{S}(\kappa)$ be the set of isomorphism classes of separated schemes of finite type over κ. For a separated scheme U of finite type over κ, we denote by $[U]$ the element of $\mathbf{S}(\kappa)$ defined by U. For a function $f\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$, we denote $f([U])$ by $f(U)$ for simplicity of notation. Let r be a nonnegative integer. Then Serre has conjectured that there exists a unique function

$$h^r_{\mathrm{c}}\colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

satisfying the following two properties (see [**33**]):

▷ For a closed subscheme Z of U,

$$h^r_{\mathrm{c}}(U) = h^r_{\mathrm{c}}(Z) + h^r_{\mathrm{c}}(U \setminus Z). \tag{I-a}$$

▷ If U is a proper smooth scheme over κ, then

$$h^r_{\mathrm{c}}(U) = (-1)^r \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U, \mathbb{Q}_\ell) \tag{I-b}$$

for a prime number $\ell \neq p$.

Here the letter 'c' in the notation h^r_{c} stands for the compact support. Heuristically $h^r_{\mathrm{c}}(U)$ is given by the formula (see [*loc. cit.*]):

$$h^r_{\mathrm{c}}(U) = \begin{cases} \sum_{m=0}^{\infty} (-1)^m \dim_{\mathbb{Q}_\ell} \mathrm{gr}^P_r H^m_{\text{ét},c}(U, \mathbb{Q}_\ell) & (p > 0), \\ \sum_{m=0}^{\infty} (-1)^m \dim_{\mathbb{Q}} \mathrm{gr}^P_r H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}), \end{cases} \tag{1.0.1}$$

where P is the 'weight filtration' on $H^m_{\text{ét},c}(U, \mathbb{Q}_\ell)$ (resp. $H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q})$). (For the cohomology without compact support, see (1.0.3) below for the definition of the weight filtration P when κ is $\overline{\mathbb{F}}_p$ (resp. $\mathbb{C}$).) Set

$$H^m_{\mathrm{c}}(U) = \begin{cases} H^m_{\text{ét},c}(U, \mathbb{Q}_\ell) & (p > 0), \\ H^m_{\mathrm{c}}(U_{\mathrm{an}}, \mathbb{Q}) & (p = 0, \kappa = \mathbb{C}). \end{cases}$$

Heuristically (I-a) is obtained from the strict exactness with respect to P of the exact sequence

$$\cdots \longrightarrow H^m_{\mathrm{c}}(U \setminus Z) \longrightarrow H^m_{\mathrm{c}}(U) \longrightarrow H^m_{\mathrm{c}}(Z) \longrightarrow \cdots .$$

Serre has called $h^r_{\mathrm{c}}(U)$ the virtual Betti number of $[U]$. He has also remarked that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^r_{c}. Serre's conjecture was the starting point of Grothendieck's theory of weights (see [**33**]).

In [**33**] Grothendieck has conjectured, for two nonnegative integers i and j, there exists a function

$$h^{ij}_{\mathrm{c}} \colon \mathbf{S}(\kappa) \longrightarrow \mathbb{Z}$$

such that

$$h^r_{\mathrm{c}}(U) = \sum_{i+j=r} h^{ij}_{\mathrm{c}}(U). \tag{II-a}$$

In the case $p>0$, let $\mathcal{W}$ be the Witt ring of κ, K_0 the fraction field of $\mathcal{W}$ and $\mathcal{W}\Omega^i_U$ the de Rham-Witt sheaf of U of degree i (see [**47**]). We call $h^{ij}_{\mathrm{c}}(U)$ the *virtual slope number* and the *virtual Hodge number* of $[U]$ if $p>0$ and $p=0$, respectively, because we require that h^{ij}_{c} satisfies the equation:

$$h^{ij}_{\mathrm{c}}(U) = \begin{cases} (-1)^{i+j}\dim_{K_0}(H^j(U,\mathcal{W}\Omega^i_U)\otimes_{\mathcal{W}} K_0) & (p>0), \\ (-1)^{i+j}\dim_\kappa H^j(U,\Omega^i_{U/\kappa}) & (p=0) \end{cases} \tag{II-b}$$

if U is a proper smooth scheme over κ. We also require that h^{ij}_{c} satisfies the equation

$$h^{ij}_{\mathrm{c}}(U) = h^{ij}_{\mathrm{c}}(Z) + h^{ij}_{\mathrm{c}}(U\setminus Z). \tag{II-c}$$

Furthermore, we conjecture that h^{ij}_{c} is uniquely determined by the equations (II-b) and (II-c). It is easy to check that the (conjectural) resolution of singularities (in the positive characteristic case) immediately implies the uniqueness of h^{ij}_{c}. The existence of h^{ij}_{c} $(i,j\in\mathbb{N})$ implies the existence of h^r_{c} $(r\in\mathbb{N})$ because (I-a) is clear by (II-a) and (II-c) and because (I-b) holds as follows: if U is a proper smooth scheme over κ, then we have the equations

$$\begin{aligned} \dim_{\mathbb{Q}_\ell} H^r_{\text{ét}}(U,\mathbb{Q}_\ell) &= \begin{cases} \dim_{K_0}(H^r_{\mathrm{crys}}(U/\mathcal{W})\otimes_{\mathcal{W}} K_0) & (p>0), \\ \dim_\kappa H^r_{\mathrm{dR}}(U/\kappa) & (p=0), \end{cases} \\ &= \begin{cases} \sum_{i+j=r}\dim_{K_0}(H^j(U,\mathcal{W}\Omega^i_U)\otimes_{\mathcal{W}} K_0) & (p>0), \\ \sum_{i+j=r}\dim_\kappa H^j(U,\Omega^i_{U/\kappa}) & (p=0). \end{cases} \end{aligned} \tag{1.0.2}$$

The first (resp. the second) equation in (1.0.2) in the case $p>0$ has been obtained in [**54**], [**16**] and [**69**] (resp. [**47**]). The first equation in (1.0.2) for