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OVERCONVERGENT MODULAR SYMBOLS AND p-ADIC L-FUNCTIONS

BY ROBERT POLLACK AND GLENN STEVENS

ABSTRACT. – This paper is a constructive investigation of the relationship between classical modular symbols and overconvergent p-adic modular symbols. Specifically, we give a constructive proof of a *control theorem* (Theorem 1.1) due to the second author [19] proving existence and uniqueness of overconvergent eigenliftings of classical modular eigensymbols of *non-critical slope*. As an application we describe a polynomial-time algorithm for explicit computation of associated p-adic L-functions in this case. In the case of *critical slope*, the control theorem fails to always produce eigenliftings (see Theorem 5.14 and [16] for a salvage), but the algorithm still "succeeds" at producing p-adic L-functions. In the final two sections we present numerical data in several critical slope examples and examine the Newton polygons of the associated p-adic L-functions.

RÉSUMÉ. – Cet article est une exploration constructive des rapports entre les symboles modulaires classiques et les symboles modulaires p-adiques surconvergents. Plus précisément, nous donnons une preuve constructive d'un théorème de contrôle (Théorème 1.1) du deuxième auteur [19] ; ce théorème démontre l'existence et l'unicité des « liftings propres » des symboles propres modulaires classiques de pente non-critique. Comme application, nous décrivons un algorithme en temps polynomial pour le calcul explicite des fonctions L p-adiques associées dans ce cas-là. Dans le cas de pente critique, le théorème de contrôle échoue toujours à produire des « liftings propres » (voir Théorème 5.14 et [16] pour un succédané), mais l'algorithme « réussit » néanmoins à produire des fonctions L p-adiques. Dans les deux dernières sections, nous présentons des données numériques pour plusieurs exemples de pente critique et examinons le polygone de Newton des fonctions L p-adiques associées.

1. Introduction

Fix a prime p, and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ denote a congruence subgroup of level prime to p. For k a non-negative integer, let $\mathcal{D}_k(\mathbb{Z}_p)$ denote the space of locally analytic p-adic distributions

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on $\mathbb{Z}_p$ endowed with the weight k action(1) of $\Gamma_0 := \Gamma \cap \Gamma_0(p)$. In [19], the second author introduced the space of overconvergent modular symbols, $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$; this is the space of Γ_0-equivariant maps from $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ to $\mathcal{D}_k(\mathbb{Z}_p)$:

$$\left\{\Phi : \Delta_0 \to \mathcal{D}_k(\mathbb{Z}_p) \ : \ \Phi(\gamma D) = \Phi(D)\big|\gamma^{-1} \text{ for } \gamma \in \Gamma_0,\ D \in \Delta_0\right\}.$$

The space $\mathcal{D}_k(\mathbb{Z}_p)$ admits a surjective map to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$ inducing a Hecke-equivariant map

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathrm{Sym}^k(\mathbb{Q}_p^2)),$$

which we refer to as the *specialization map*.

The target of this map is the space of classical modular symbols of level Γ_0 and weight k. By Eichler-Shimura theory, this space is finite-dimensional and contains all systems of Hecke-eigenvalues which occur in the space of classical modular forms of weight $k+2$ and level Γ_0. On the other hand, the source of the specialization map $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))$ is certainly an infinite-dimensional space. Nonetheless, in [19], the following control theorem is given for the specialization map, which can be viewed as an analogue of Coleman's theorem on small slope overconvergent forms being classical (see also [16]).

Theorem 1.1. – *The specialization map restricted to the subspace where U_p acts with slope strictly less that $k+1$*

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{(<k+1)} \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathrm{Sym}^k(\mathbb{Q}_p^2))^{(<k+1)}$$

is a Hecke-equivariant isomorphism.

The main goal of this paper is to provide an explicit enough description of these spaces of overconvergent modular symbols to:

1. give a constructive proof of the above theorem (see Theorem 5.12), and
2. perform explicit computations in these spaces, constructing overconvergent Hecke-eigensymbols.

1.1. Explicit description of modular symbols

Our first step in explicitly constructing overconvergent modular symbols is to give a simple description of $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ as a left $\mathbb{Z}[\Gamma]$-module in terms of generators and relations for Γ, any finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

In the case when Γ is torsion-free (which we will assume for the remainder of the introduction), we will be able to find divisors $D_1, D_2, \ldots, D_t \in \Delta_0$ such that together with $D_\infty := \{\infty\} - \{0\}$, they generate Δ_0, and satisfy the single relation

$$\left(\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) - 1\right) D_\infty = \sum_{i=1}^{t} (1 - \gamma_i^{-1}) D_i$$

for some $\gamma_i \in \Gamma$ (see Theorem 2.6).

To "solve the Manin relations" in such a way that we have only a single relation, we make use of a fundamental domain for the action of Γ on the upper-half plane with a particularly

(1) In this paper, we will consider *right* actions on $\mathcal{D}_k(\mathbb{Z}_p)$; moreover, we normalize our weights so that weight 0 corresponds to the trivial action (and thus to weight 2 modular forms).

nice shape; we form a fundamental domain whose interior is connected, and whose boundary is a union of unimodular paths.

Having this single relation in hand, one can construct a symbol $\varphi \in \mathrm{Symb}_\Gamma(V)$, where V is an arbitrary right $\mathbb{Z}[\Gamma]$-module, by producing elements $v_1, \ldots, v_t, v_\infty$ in V satisfying

$$v_\infty \big| \Delta = \sum_{i=1}^{t} v_i \big| (1 - \gamma_i)$$

where $\Delta = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) - 1$. One simply sets $\varphi(D_i) = v_i$, and then extends to all of Δ_0.

Note that to be able to write down such elements in V, one needs to be able to solve an equation of the form

$$v_\infty \big| \Delta = w,$$

which we will refer to as the *difference equation*.

1.2. Distributions, moments, and the difference equation

The span of the functions $\{z^j\}_{j=0}^{\infty}$ is dense in the space of locally analytic functions on $\mathbb{Z}_p$. In particular, a distribution $\mu \in \mathcal{D}_k(\mathbb{Z}_p)$ is uniquely determined by the values $\{\mu(z^j)\}_{j=0}^{\infty}$, its sequence of moments.

Using this description of a distribution by its sequence of moments, one can write down an explicit solution to the difference equation in these spaces of distributions. Specifically, let ν_j be the simple distribution defined by

$$\nu_j(z^r) = \begin{cases} 1 & \text{if } r = j \\ 0 & \text{otherwise,} \end{cases}$$

and let $\eta_j \in \mathcal{D}_k(\mathbb{Z}_p)$ be defined by

$$\eta_j(z^r) = \begin{cases} \binom{r}{j} b_{r-j} & \text{if } r \geq j \\ 0 & \text{otherwise,} \end{cases}$$

where b_n is the n-th Bernoulli number. We then have (see Lemma 4.4)

$$\frac{\eta_j \big| \Delta}{j+1} = \nu_{j+1}.$$

One obtains a general solution of the difference equation by combining these basic solutions.[2] It is interesting that Bernoulli numbers play such a key role in this solution.

[2] The denominators that appear in this solution could potentially cause a problem. It is primarily for this reason that in the text of the paper we work with $\mathcal{D}_k^\dagger$, a larger space of distributions.

1.3. Sketch of a proof of Theorem 1.1

Let φ in $\mathrm{Symb}_{\Gamma_0}(\mathrm{Sym}^k(\mathbb{Q}_p^2))$ be an eigensymbol of slope strictly less than $k+1$. By combining our explicit solution of the Manin relations together with our solution of the difference equation, we can explicitly compute an overconvergent modular symbol Ψ lifting φ. To produce an *eigensymbol* lifting φ, we consider the sequence

$$\Psi_n := \frac{\Psi \big| U_p^n}{\lambda^n}$$

where λ is the U_p-eigenvalue of φ. Since the specialization map is Hecke-equivariant, Ψ_n also lifts φ. Then, using the fact that the valuation of λ is strictly less than $k+1$, we show that $\{\Psi_n\}$ converges to some symbol Φ, which still lifts φ. By construction, Φ is a U_p-eigensymbol with eigenvalue λ, and is thus the symbol we are seeking.

1.4. Connection with p-adic L-functions

We note that overconvergent Hecke-eigensymbols are directly related to p-adic L-functions. Indeed, let f be a classical modular form on Γ_0 of non-critical slope (*i.e.* with slope strictly less than $k+1$), and let φ_f denote its corresponding classical modular symbol. As f is assumed to have non-critical slope, by Theorem 1.1, there is a unique symbol Φ_f in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{(<k+1)}$ which specializes to φ_f. It is proven in [19] that

$$\Phi_f(\{\infty\} - \{0\}) = L_p(f),$$

the p-adic L-function of f (see also Theorem 6.3 in this paper).

Thus, the above proof of Theorem 1.1 can be converted into an algorithm for computing p-adic L-functions. Indeed, starting with a modular form f, using standard methods, one can produce its corresponding modular symbol φ_f. As described above, one lifts φ_f to an overconvergent symbol, and then iterates the operator U_p/λ on this lift. If f is a p-ordinary modular form, then each application of U_p yields an extra digit of p-adic accuracy. Evaluating at the divisor $\{\infty\} - \{0\}$ then gives an approximation to the p-adic L-function.

We note that computing p-adic L-functions of modular forms from their definition (i.e. by using Riemann sums) requires evaluating a sum with around p^n terms in order to get n digits of p-adic accuracy. Such an algorithm is exponential in n, and thus cannot be used to compute these L-functions to high accuracy even for small p. The above algorithm is polynomial in n (see [6, Proposition 2.14]), and can be used in practice to compute p-adic L-functions to very high accuracy.

1.5. Approximating distributions

We now describe a systematic method of approximating distributions which one needs in order to carry out the algorithm described above.

Recall that $\mu \in \mathcal{D}_k(\mathbb{Z}_p)$ is uniquely determined by its sequence of moments $\{\mu(z^j)\}$ for $j \geq 0$. To approximate such a distribution, one might hope to keep track of the first M moments, each modulo p^N for some integers $M, N \gg 0$. Unfortunately, this approach cannot be used for our purposes because such approximations are not stable under the matrix actions we are considering; that is, if one has the above data for a distribution μ, one