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L^p ESTIMATES FOR MULTI-LINEAR AND MULTI-PARAMETER PSEUDO-DIFFERENTIAL OPERATORS

BY WEI DAI & GUOZHEN LU

ABSTRACT. — We establish the pseudo-differential variant of the L^p estimates for multi-linear and multi-parameter Coifman-Meyer multiplier operators proved by C. Muscalu, J. Pipher, T. Tao and C. Thiele in [21, 22]. This gives an affirmative answer to the question, raised in the book of C. Muscalu and W. Schlag [23], on whether the L^p estimates for multi-linear and multi-parameter pseudo-differential operators hold.

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1. Introduction

1.1. Background. — For $n \geq 1$ and $d \geq 1$, let m be a bounded function in $\mathbb{R}^{nd}$, smooth away from the origin and satisfying Hörmander-Mikhlin conditions ⁽¹⁾

$$(1.1) \quad |\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}$$

for sufficiently many multi-indices α . Denote by T_m the n -linear operator defined by

$$(1.2) \quad T_m(f_1, \dots, f_n)(x) := \int_{\mathbb{R}^{nd}} m(\xi) \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_n)} d\xi,$$

where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^{nd}$ and $f_1, \dots, f_n$ are Schwartz functions on $\mathbb{R}^d$. From the classical Coifman-Meyer theorem (see [6, 7, 19, 11, 15]), we know that the operator T_m extends to a bounded n -linear operator from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_n}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, provided that $1 < p_1, \dots, p_n \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} > 0$. When $n = 2$, as a consequence of bilinear $T1$ theorem (see [6, 11]), there is also a pseudo-differential variant of the classical Coifman-Meyer theorem for symbol $a \in BS_{1,0}^0(\mathbb{R}^{3d})$, that is, a satisfies the differential inequalities

$$(1.3) \quad |\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta a(x, \xi, \eta)| \lesssim_{d,\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^{-|\alpha| - |\beta|}$$

for sufficiently many multi-indices α, β, γ . Namely, let T_a be the corresponding bilinear pseudo-differential operators defined by replacing m with a in (1.2), then T_a is bounded from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$, provided that $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$ (see [2], and see [3, 26, 23] for $d = 1$ case). For large amounts of literature involving estimates for multi-linear Calderón-Zygmund operators and multi-linear pseudo-differential operators, refer to e.g., [1, 6, 19, 9, 11, 12, 15, 23, 24].

However, when we come into the situation that a differential operator (with different behaviors on different spatial variables x_i , $i = 1, \dots, d$) acts on a product of several functions (for instance, the bilinear form $\mathcal{D}_1^\alpha \mathcal{D}_2^\beta(fg)$, where $\widehat{\mathcal{D}_1^\alpha f(\xi_1, \xi_2)} := |\xi_1|^\alpha \hat{f}(\xi_1, \xi_2)$ and $\widehat{\mathcal{D}_2^\beta f(\xi_1, \xi_2)} := |\xi_2|^\beta \hat{f}(\xi_1, \xi_2)$ for $\alpha, \beta > 0$), we realize that the necessity to investigate bilinear and bi-parameter operators

⁽¹⁾ Throughout this paper, $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. If necessary, we use explicitly $A \lesssim_{*,\dots,*} B$ to indicate that there exists a positive constant $C_{*,\dots,*}$ depending only on the quantities appearing in the subscript continuously such that $A \leq C_{*,\dots,*} B$.

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$T_m^{(2)}$ defined by

$$(1.4) \quad T_m^{(2)}(f, g)(x) := \int_{\mathbb{R}^4} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta,$$

where the symbol m is smooth away from the planes $(\xi_1, \eta_1) = (0, 0)$ and $(\xi_2, \eta_2) = (0, 0)$ in $\mathbb{R}^2 \times \mathbb{R}^2$ and satisfying the less restrictive Marcinkiewicz conditions

$$(1.5) \quad |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\beta_1} \partial_{\eta_1}^{\alpha_2} \partial_{\eta_2}^{\beta_2} m(\xi, \eta)| \lesssim \frac{1}{|(\xi_1, \eta_1)|^{\alpha_1 + \alpha_2}} \cdot \frac{1}{|(\xi_2, \eta_2)|^{\beta_1 + \beta_2}}$$

for sufficiently many multi-indices $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. It becomes more complicated and difficult to establish the L^p estimates for $T_m^{(2)}$ than in the one-parameter multilinear situations or L^p estimates for linear multi-parameter singular integrals (see e.g., [8] and [14]). In [21], by using the duality lemma of $L^{p, \infty}$ presented in [24], the $L^{1, \infty}$ sizes and energies technique developed in [25] and multi-linear interpolation (see e.g., [10, 25]), Muscalu, Pipher, Tao and Thiele proved the following L^p estimates for $T_m^{(2)}$ (see also [23], and for subsequent endpoint estimates see [16]).

THEOREM 1.1 ([21]). — *The bilinear operator $T_m^{(2)}$ defined by (1.4) maps $L^p(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \rightarrow L^r(\mathbb{R}^2)$ boundedly, as long as $1 < p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$.*

In general, any collection of n generic vectors $\xi_1 = (\xi_1^i)_{i=1}^d, \dots, \xi_n = (\xi_n^i)_{i=1}^d$ in $\mathbb{R}^d$ generates naturally the following collection of d vectors in $\mathbb{R}^n$:

$$(1.6) \quad \bar{\xi}_1 = (\xi_j^1)_{j=1}^n, \quad \bar{\xi}_2 = (\xi_j^2)_{j=1}^n, \quad \dots, \quad \bar{\xi}_d = (\xi_j^d)_{j=1}^n.$$

Let $m = m(\xi) = m(\bar{\xi})$ be a bounded symbol in $L^\infty(\mathbb{R}^{dn})$ that is smooth away from the subspaces $\{\bar{\xi}_1 = 0\} \cup \dots \cup \{\bar{\xi}_d = 0\}$ and satisfying

$$(1.7) \quad |\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} m(\bar{\xi})| \lesssim \prod_{i=1}^d |\bar{\xi}_i|^{-|\alpha_i|}$$

for sufficiently many multi-indices $\alpha_1, \dots, \alpha_d$. Denote by $T_m^{(d)}$ the n -linear multiplier operator defined by

$$(1.8) \quad T_m^{(d)}(f_1, \dots, f_n)(x) := \int_{\mathbb{R}^{dn}} m(\xi) \hat{f}_1(\xi_1) \dots \hat{f}_n(\xi_n) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_n)} d\xi.$$

In [22], Muscalu, Pipher, Tao and Thiele generalized Theorem 1.1 to the n -linear and d -parameter setting for any $n \geq 1$, $d \geq 2$, their result is stated in the following theorem (see also [23]).