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SMOOTHING LOW-DIMENSIONAL ALGEBRAIC CYCLES
[*after* Kollár and Voisin]

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1. Introduction

1.1. The smoothing problem for algebraic cycles

Let k be a field of characteristic 0 (which the reader may choose to be the field $\mathbb{C}$ of complex numbers). Let X be a smooth projective algebraic variety of dimension n over k. A central theme in algebraic geometry is the study of algebraic cycles on X, that is of the collection of all the algebraic subvarieties of X. We will always denote by d the dimension of the subvarieties we consider.

The main players of this line of research are the Chow groups $\mathrm{CH}_d(X)=\mathrm{CH}^{n-d}(X)$ of X. Their elements are *cycles*: linear combinations with integral coefficients of (closed, integral) d-dimensional algebraic subvarieties of X, considered up to *rational equivalence*. Here, two cycles are said to be rationally equivalent if both belong to the same algebraic family of cycles on X parametrized by the projective line $\mathbb{P}^1_k$. We refer to [8] for precise definitions of Chow groups [8, §1.3], for their functorial properties [8, §1.4, §1.7 and Chap. 6] and for the construction of the intersection product on them [8, §8.3].

One should think of these groups as algebro-geometric analogues of singular homology groups, where both the generators and the relations are constrained to have an algebraic (as opposed to topological) origin. No conditions are imposed on the algebraic subvarieties of X that generate its Chow groups; in particular, they may carry arbitrary singularities. The next question, first asked by Borel and Haefliger in [4, §5.17], is therefore of interest.

Question 1.1. *Let X be a smooth projective algebraic variety over k. Are the Chow groups $\mathrm{CH}_*(X)$ of X generated by classes of smooth subvarieties of X?*

In other words, is it possible to smooth algebraic cycles up to rational equivalence? To be precise, the original question of Borel and Haefliger was slightly weaker.

They considered, over the field $k = \mathbb{C}$, the coarser *homological equivalence* relation, where two algebraic cycles are identified if they have the same image by the cycle class map $\mathrm{CH}_d(X) \to H_{2d}(X, \mathbb{Z})$.

One of the goals of this introduction is to review the state of the art concerning Question 1.1 (see §1.4 for positive results and §1.5 for negative results).

1.2. Smoothing algebraic cycles in the Whitney range

Let us state right away the main theorem presented in this survey, which has been recently proved by Kollár and Voisin [15, Theorem 1.2].

Theorem 1.2. *Let X be a smooth projective algebraic variety of dimension n over k. If $d < \frac{n}{2}$, then $\mathrm{CH}_d(X)$ is generated by classes of smooth subvarieties of X.*

At the time when Borel and Haefliger asked Question 1.1, resolution of singularities was not available. A positive answer to this question could therefore have been used as a substitute to resolution of singularities in the study of algebraic cycles. This original motivation has nowadays disappeared. On the contrary, the proof of Theorem 1.2 does use Hironaka's theorem [10] on resolution of singularities (exactly once, in the proof of Theorem 1.7 given at the end of §1.3).

It is verified in [14, Theorem 39] that the proof of Theorem 1.2 can be extended to base fields k of characteristic $p \geqslant n - d$, at least if k is assumed to be infinite and perfect. In this argument, resolution of singularities is replaced with Gabber's improvement [11, Theorem 2.1] of de Jong's alteration theorem.

Let us explain the significance and the importance of the restrictive hypothesis $d < \frac{n}{2}$ in the statement of Theorem 1.2. It plays the exact same role as in Whitney's weak embedding theorem [20] in differential topology (according to which any compact $\mathscr{C}^\infty$ manifold of dimension d embeds in $\mathbb{R}^n$ if $d < \frac{n}{2}$). The heuristic in our algebraic situation is that a morphism $f\colon Y \to X$ of smooth projective varieties over k which is sufficiently generic is an embedding if $\dim(Y) < \frac{n}{2}$. Its image $f(Y)$ is then a smooth subvariety of X. This suggests that, under the hypothesis $d < \frac{n}{2}$, the variety X should contain many smooth subvarieties constructed by general projection arguments, therefore increasing the likelihood that $\mathrm{CH}_d(X)$ is generated by classes of smooth subvarieties of X. Of course, the difficulty is to enforce this genericity condition on f by algebraic means.

A Whitney-type hypothesis had already appeared in two earlier works on Question 1.1. On the one hand, Hironaka had given a proof of Theorem 1.2 under the additional assumption that $d \leqslant 3$ (see [10, Theorem p. 50]). On the other hand, I had constructed counterexamples to Question 1.1 on the boundary $d = \frac{n}{2}$ of the Whitney range, for infinitely many values of d (see [1, Theorem 0.3]), thereby showing that the hypothesis $d < \frac{n}{2}$ in Theorem 1.2 is optimal (for these values of d). These works will be discussed in more details in §1.4.1 and §1.5.2 respectively.

1.3. Flat pushforwards of complete intersections

The approach of Kollár and Voisin relies chiefly on the following definition. Recall that a morphism $f\colon Y \to X$ of connected smooth projective varieties over k is *flat* if and only if it is equidimensional, i.e. if all its fibers have dimension $\dim(Y) - \dim(X)$ (in this survey, this can be taken as a definition; see also [7, Theorem 18.16 b]).

Definition 1.3. Let X be a smooth projective variety over k. Define $\mathrm{CH}_d(X)_{\mathrm{KV}}$ to be the subgroup of $\mathrm{CH}_d(X)$ generated by cycles of the form $f_*(\lambda_1 \dots \lambda_c)$ for some flat morphism $f\colon Y \to X$ of smooth projective varieties over k and some codimension 1 classes $\lambda_1, \dots, \lambda_c \in \mathrm{CH}^1(Y)$, where $c + d = \dim(Y)$.

In short, the subgroup $\mathrm{CH}_d(X)_{\mathrm{KV}} \subset \mathrm{CH}_d(X)$ is generated by those cycles that may be written as flat pushforwards of intersections of divisor classes. Theorem 1.2 results from the combination of the following two theorems.

Theorem 1.4 ([15, Proposition 1.5])**.** *Let X be a smooth projective variety of dimension n over k. If $d < \frac{n}{2}$, then any element of $\mathrm{CH}_d(X)_{\mathrm{KV}}$ may be written as a linear combination with integral coefficients of classes of smooth subvarieties of X.*

Theorem 1.5 ([15, Theorem 1.6])**.** *If X is a smooth projective variety over k, then*

$$\mathrm{CH}_*(X)_{\mathrm{KV}} = \mathrm{CH}_*(X).$$

Theorem 1.4 is the part of the proof of Theorem 1.2 where the smoothing actually takes place. It is also the part in which the Whitney-type hypothesis $d < \frac{n}{2}$ is used. It is directly inspired by Hironaka's work [10] on the topic. We discuss it more in §1.4.1, and present its proof in Section 2.

In contrast, Theorem 1.5 holds with no restriction on the dimension of the cycles. It constitutes a fundamentally new structural result on the Chow groups of arbitrary smooth projective varieties. Its formulation and its proof are the main achievements of the article [15]. It is not known whether Theorem 1.5 would still hold if one required the morphisms f in Definition 1.3 to be smooth instead of only flat (see [15, Question 1.11]). The flexibility gained by allowing flat morphisms that are possibly not smooth is used exactly once in the proof of Theorem 1.5 (in the proof of Proposition 3.1).

As noted in [14], the proof of Theorem 1.5 given in [15] yields a slightly stronger result, valid at the level of subvarieties (as opposed to Chow groups). To state it, we rely on the next geometric definition. Recall that a subvariety of codimension c in a smooth projective variety X over k is said to be a *complete intersection* if it can be written as the intersection of c hypersurfaces in X.

Definition 1.6. Let X be a smooth projective variety over k. An integral subvariety $Z \subset X$ is said to be a smooth complete intersection image (or *sci-image* for short) if there exist a flat morphism $f\colon Y \to X$ of smooth projective varieties over k and a smooth complete intersection $V \subset Y$ such that $f(V) = Z$ and $f|_V\colon V \to Z$ is birational.

Theorem 1.7 ([14, Theorem 2])**.** *Let X be a smooth projective variety over k. Any integral subvariety $Z \subset X$ is an sci-image.*

Theorem 1.5 is an immediate consequence of Theorem 1.7. In turn, the proof of Theorem 1.7 has two steps. In the first step, one studies functorial properties of sci-images. The culmination of this analysis is the next proposition. In its statement, one makes use of the following terminology: a subvariety of codimension c in a smooth projective variety X over k is said to be a *complete bundle section* (or *cbs* for short) if it can be written as the zero locus of a section of a vector bundle of rank c on X.

Proposition 1.8 ([15, Proposition 3.11], [14, Lemma 16])**.** *Let $\pi\colon X' \to X$ be the blow-up of a smooth cbs in a smooth projective variety over k. Let $Z' \subset X'$ be an sci-image such that $\pi|_{Z'}\colon Z' \to Z := \pi(Z')$ is birational. Then $Z \subset X$ is an sci-image.*

To conclude the proof of Theorem 1.7, one would like to show that any integral subvariety of X can be turned into a smooth complete intersection after repeatedly blowing up smooth cbs. This is not known to hold (see e.g. [15, Question 4.1]) and the second step is a weaker statement, which is sufficient for our purposes.

Proposition 1.9 ([15, Theorem 1.9])**.** *Let X be a smooth projective variety of dimension n over k. Let $Z \subset X$ be a smooth subvariety of dimension $< \frac{n}{4}$. Then there exist a composition $\pi^+\colon X^+ \to X$ of blow-ups of smooth cbs and a smooth complete intersection $V \subset X^+$ such that $\pi^+(V) = Z$ and $\pi^+|_V\colon V \to Z$ is birational.*

Proposition 1.9 is the technical heart of [15]. Propositions 1.8 and 1.9 will be proved in Sections 3 and 4 respectively. Together, they imply Theorem 1.7, as we now explain.

Proof of Theorem 1.7. Let $\widetilde{Z} \to Z$ be a resolution of singularities (see [10]). Embed $\widetilde{Z}$ in $\mathbb{P}^N_k$ for some $N \geqslant 0$. View $\widetilde{Z}$ as a subvariety of $Y := X \times \mathbb{P}^N_k$ using the natural diagonal embedding. After possibly increasing N, one can apply Proposition 1.9 to the subvariety $\widetilde{Z}$ of Y. In this way, we obtain a composition $\pi^+\colon Y^+ \to Y$ of blow-ups of smooth cbs and a smooth complete intersection $V \subset Y^+$ such that $\pi^+(V) = \widetilde{Z}$ and $\pi^+|_V\colon V \to \widetilde{Z}$ is birational. Using Proposition 1.8, we deduce that the subvariety $\widetilde{Z} \subset Y$ is an sci-image. It follows that $Z \subset X$ is also an sci-image, by flatness of the first projection morphism $Y = X \times \mathbb{P}^N_k \to X$. □

1.4. Smoothing techniques

We now focus on the various techniques that have been used to smooth algebraic cycles up to rational equivalence, both in the early positive results about Question 1.1 and in the recent work of Kollár and Voisin. These techniques fall into two categories, depending on whether the algebraic cycles under consideration have small dimension, or small codimension. In the first case, they are best thought of homologically and presented as pushforwards (see §1.4.1). In the second case they are best thought of cohomologically and presented as pullbacks (see §1.4.2). Combining the results presented in §1.4.1 and §1.4.2 shows that Question 1.1 has a positive answer when $n \leqslant 5$.

1.4.1. *Cycles of small dimension.* — The first progress on Question 1.1 was due to Hironaka [10, Theorem p. 50] who answered it positively when $d < \frac{n}{2}$ (the Whitney-type condition discussed in §1.2) and $d \leqslant 3$. Let us sketch his proof.

Let $Z \subset X$ be an integral subvariety of dimension d, and let $\widetilde{Z} \to Z$ be a resolution of singularities. Choosing an embedding of $\widetilde{Z}$ in $\mathbb{P}^N_k$ for some $N \geqslant 0$ allows one to view $\widetilde{Z}$ as a subvariety of $Y := X \times \mathbb{P}^N_k$. (That Hironaka's argument inspired the proof of Theorem 1.7 given at the end of §1.3 should be obvious.) At that point, the idea is to find a cycle $\widetilde{Z}'$ in Y, which is rationally equivalent to $\widetilde{Z}$, and whose components are smooth and in general position. The image Z' of $\widetilde{Z}'$ in X is then rationally equivalent to Z, and has smooth components precisely because of the Whitney-type hypothesis.

To construct $\widetilde{Z}'$ from $\widetilde{Z}$, one needs some kind of moving lemma. To this effect, Hironaka devises a *moving by linkage* technique. Let $\mathscr{L}$ be a sufficiently ample line bundle on Y, and let c be the codimension of $\widetilde{Z}$ in Y. Let $D_1, \dots, D_c$ be general elements of the linear system $|\mathscr{L}|$, and let $E_1, \dots, E_c$ be general elements of $|\mathscr{L}|$ containing $\widetilde{Z}$. One can write $E_1 \cap \cdots \cap E_c = \widetilde{Z} \cup W$ for some subvariety $W \subset Y$ of codimension c. One says that the subvarieties $\widetilde{Z}$ and W of Y are *linked*. Our choices ensure that the cycle $\widetilde{Z}' := (D_1 \cap \cdots \cap D_c) - W$ is rationally equivalent to $\widetilde{Z}$. The subvariety W is in general singular in codimension 4, so it is smooth when $d \leqslant 3$. To complete the proof, repeat the linkage procedure a few times to enforce the general position hypothesis.

For cycles of dimension $\geqslant 4$, Hironaka's method fails because of the singularities that linkage inevitably creates. However, combined with additional arguments to control these singularities, these ideas may still be useful (see [1, Theorems 0.4 and 0.6] for applications to real algebraic cycles).

The perspective of Kollár and Voisin is very different. They do not attempt to control the singularities that appear. Neither do they try to develop another moving technique applicable to general cycles. Instead, they prove the structural result that all cycles come by flat pushforward from complete intersections (Theorem 1.5) and