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UPPER BOUNDS ON DIAGONAL RAMSEY NUMBERS
[*after* Campos, Griffiths, Morris, and Sahasrabudhe]

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1. Introduction

Ramsey theory is a branch of combinatorics that studies order and disorder. The underlying mantra of the field, as articulated by Theodore Motzkin, is that "complete disorder is impossible"—any sufficiently large system must have a large, highly structured subsystem. The prototypical example of a Ramsey-theoretic statement is *Ramsey's theorem*, from which the field derives its name.

Theorem 1.1 (Ramsey, 1929). *For every integer $k \geqslant 2$, there exists some positive integer N such that any two-coloring of the edges of the complete graph*[(1)] *K_N contains a monochromatic K_k.*

In other words, no matter how we assign the edges of K_N a color, say red or blue, we can always find k vertices such that all edges between them receive the same color. That is, any such coloring, no matter how unstructured, contains a highly structured subcoloring. Even this simple statement has some remarkable consequences. For example, Schur (1917) used Theorem 1.1[(2)] to prove that for all sufficiently large primes p, there exist non-trivial solutions to the equation $x^n + y^n \equiv z^n \pmod{p}$, that is, that one cannot prove Fermat's last theorem via a local-global argument.

Connections and applications to other fields of mathematics have been an important feature of Ramsey theory from the very beginning. Ramsey himself had an application in mathematical logic in mind when he proved Theorem 1.1 (indeed, his paper is titled "On a problem of formal logic"). The influential paper of Erdős and Szekeres (1935), which helped establish Ramsey theory as a central branch of combinatorics, is titled "A combinatorial problem in geometry"; in it, they reproved Theorem 1.1 in order to deduce a result on convex polygons among sets of points in Euclidean space.

(1) Recall that the complete graph K_N has N vertices, and all of the $\binom{N}{2}$ possible edges are present.
(2) Alert readers may note that Schur's result precedes Ramsey's by more than a decade. In fact, Schur proved a closely related lemma, which one can now recognize as a consequence of Theorem 1.1, and derived his theorem from that lemma.

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Today, Ramsey-theoretic theorems and techniques are of fundamental importance in many different fields, including additive number theory, Banach space theory, discrete geometry, ergodic theory, group theory, and theoretical computer science. These are deep and rich connections, and are difficult to adequately summarize, so we refer to the book of Graham, Rothschild, and Spencer (1990), to the survey of Conlon, Fox, and Sudakov (2015), and to the lecture notes of Wigderson (2024) for more in-depth introductions to the field.

For many applications, such as those of Schur (1917) in number theory, Ramsey (1929) in logic, and Erdős and Szekeres (1935) in geometry mentioned above, qualitative statements such as Theorem 1.1 suffice. However, much of the modern research in Ramsey theory is concerned with *quantitative* statements: how large is the integer N in Theorem 1.1 as a function of k? Formally, we make the following definition.

Definition 1.2. The *Ramsey number* $r(k)$ is the least integer N such that every two-coloring of the edges of K_N contains a monochromatic K_k.

Before continuing with the discussion of what is known about the function $r(k)$, let us pause and ask why we should study such quantitative questions, when qualitative statements like Theorem 1.1 are elegant and already suffice for many applications. There are several answers to this question. One answer is that for certain applications, especially in fields such as theoretical computer science (e.g. the lower bound of Razborov (1985) on monotone circuit complexity), qualitative statements are not sufficient, as the application itself is quantitative. A second answer is that a better quantitative understanding of Ramsey-theoretic results can yield new insights and new proofs of existing theorems. For example, recent breakthroughs on the quantitative aspects of the Ramsey-theoretic theorem of Roth (1953), due to Bloom and Sisask (2020) and Kelley and Meka (2023) (see also the exposé of Peluse (2022)), imply that the primes contain infinitely many three-term arithmetic progressions. This result was first proved by van der Corput (1939), and is a special case of the landmark result of Green and Tao (2008). However, in contrast to these earlier proofs, we now know that the primes contain infinitely many three-term arithmetic progressions simply because *there are many prime numbers*. That is, the quantitative improvements yielded a new proof of this theorem, using essentially no properties of the primes other than their density. Finally, and no less importantly, a third reason for studying such quantitative questions is that doing so can reveal a world of deep and beautiful mathematics.

With that said, let us turn to the quantitative aspects of Theorem 1.1, that is, to the determination of the function $r(k)$ from Definition 1.2. The exact value of $r(k)$ is only known for $k \leqslant 4$, and it currently seems completely hopeless[(3)] to obtain an exact formula for $r(k)$, so let us content ourselves with asymptotic bounds as $k \to \infty$.

[(3)] The following famous anecdote was reported by Spencer (1994): "Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $r(5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $r(6)$. In that case, he believes, we should

Essentially every proof of Theorem 1.1 yields (at least implicitly) an upper bound on $r(k)$, by proving the existence of *some* integer N. The original proof of Ramsey (1929) gave a bound of $r(k) \leqslant k!$, but Ramsey wrote "I have little doubt that [this upper bound is] far larger than is necessary". Indeed, a few years later, Erdős and Szekeres (1935) proved the following stronger bound.

Theorem 1.3 (Erdős and Szekeres, 1935). *$r(k) \leqslant 4^k$ for every $k \geqslant 2$.*

For about a decade, it was believed that this bound was also far larger than is necessary, namely that $r(k)$ should grow subexponentially as a function of k. However, Erdős (1947) dispelled this belief by proving[(4)] an exponential lower bound.

Theorem 1.4 (Erdős, 1947). *$r(k) \geqslant \sqrt{2}^k$ for every $k \geqslant 2$.*

After this breakthrough, progress stalled for 75 years. There were a number of improvements to these bounds over the years, including important results of Spencer (1975), Graham and Rödl (1987), Thomason (1988), Conlon (2009), and Sah (2023), but all of these improvements only affected the lower-order terms, and did not improve either of the exponential constants $\sqrt{2}$ and 4. This impasse finally ended with a breakthrough of Campos, Griffiths, Morris, and Sahasrabudhe (2023).

Theorem 1.5 (Campos, Griffiths, Morris, and Sahasrabudhe, 2023). *There exists a constant $\delta > 0$ such that $r(k) \leqslant (4 - \delta)^k$ for all $k \geqslant 2$. Concretely, $r(k) \leqslant 3.993^k$ for all sufficiently large k.*

The exact constant 3.993 is not particularly important, and a more careful analysis of the same proof yields a slightly better bound[(5)]. The important thing about this result is that it is the first result, after almost 90 years of intense study, to break the barrier of 4^k.

The new tool introduced by Campos, Griffiths, Morris, and Sahasrabudhe (2023) is the so-called *book algorithm*, an elementary but ingenious technique for finding monochromatic *book graphs* in colorings of K_N. As we will shortly discuss, a book graph is a basic graph-theoretic object, whose study turns out to be closely connected to the study of Ramsey numbers. Every known proof of Theorem 1.1 uses, implicitly or explicitly, monochromatic book graphs.

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attempt to destroy the aliens." Indeed, results of Exoo (1989) and Angeltveit and McKay (2024) show that $r(5)$ takes on one of the four values $\{43, 44, 45, 46\}$, but we remain very far from knowing the value of $r(6)$.

(4) Lower bounds on Ramsey numbers are somewhat beyond the scope of this exposé, so we will not discuss the proof of Theorem 1.4 in detail. However, it would be remiss not to mention that this beautiful proof is extraordinarily influential, and is the origin of the *probabilistic method*, an extremely powerful technique in modern combinatorics.

(5) More recently, Gupta, Ndiaye, Norin, and Wei (2024) recast the proof of Theorem 1.5 in a different language, which allowed them to optimize the technique and obtain a much stronger bound of $r(k) \leqslant 3.8^k$ for sufficiently large k.

As we will see, the proof of Campos, Griffiths, Morris, and Sahasrabudhe (2023) is fairly ad hoc, and relies on the verification of certain complicated numerical inequalities. More recently, however, a new, more conceptual proof of Theorem 1.5[(6)] was found by Balister et al. (2024). They introduced a modification of the book algorithm, but their crucial new input is a purely geometric lemma, concerning the correlations of probability distributions in high-dimensional Euclidean space. While the proof of Campos, Griffiths, Morris, and Sahasrabudhe (2023) and Balister et al. (2024) have many features in common, they differ in key ways, and we will sketch both proofs.

The rest of this exposé is dedicated to discussing these two proofs of Theorem 1.5, and is organized as follows. We begin in Section 2 with a proof of Theorem 1.3, in the course of which we introduce book graphs as well as several of the key ideas that go into the proof of Theorem 1.5. In Section 3, we introduce and analyze the book algorithm of Campos, Griffiths, Morris, and Sahasrabudhe (2023), and will then *fail* to prove Theorem 1.5. Luckily, we will rescue the argument and complete the proof in Section 4 by introducing two additional ingredients. In Section 5 we introduce and analyze the symmetric book algorithm of Balister et al. (2024), and use it to give another proof of Theorem 1.5. The key new lemma introduced by Balister et al. (2024), and its geometric proof, are discussed in Section 6. We end in Section 7 with an epilogue, discussing the use of book graphs in the original proof of Ramsey (1929) of Theorem 1.1, as well as how our understanding of book graphs and Ramsey theory has developed over the subsequent 95 years.

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2. The Erdős–Szekeres theorem and algorithm

In this section, we prove Theorem 1.3 (and thus Theorem 1.1). This proof is elegant and interesting in its own right, and additionally it contains within it several of the important ideas used in the proof of Theorem 1.5. We will actually see three different proofs (or, more precisely, three different ways of viewing the same proof) of Theorem 1.3, in each of the next three subsections. Each proof will help introduce some of the key ideas that go into the proof of Theorem 1.5.

[(6)] Moreover, Balister et al. (2024) were able to prove a more general theorem, which gives a new upper bound on Ramsey numbers in any number of colors. For simplicity, however, we remain with the two-color version of the problem throughout this exposé.

2.1. Off-diagonal Ramsey numbers

We begin with the original proof of Erdős and Szekeres (1935). Before proceeding with the proof, we generalize the notion of Ramsey numbers from Definition 1.2. Here and throughout, we denote by $V(K_N)$ and $E(K_N)$ the vertex set and edge set, respectively, of the complete graph K_N.

Definition 2.1. Given integers $k, \ell \geqslant 2$, the *off-diagonal Ramsey number* $r(k,\ell)$ is the least integer N such that every two-coloring of $E(K_N)$ with colors red and blue contains a red K_k or a blue K_ℓ.

Note that $r(k,\ell) = r(\ell,k)$ as the colors play symmetric roles, and that $r(k) = r(k,k)$. The quantity $r(k)$ is often called the *diagonal Ramsey number*.

With this terminology, we can prove Theorem 1.3. In fact, we will prove the following more precise result.

Theorem 2.2 (Erdős and Szekeres, 1935). *For all integers $k, \ell \geqslant 2$, we have*

$$r(k,\ell) \leqslant \binom{k+\ell-2}{k-1}.$$

In particular,

$$r(k) \leqslant \binom{2k-2}{k-1} < 4^k.$$

Proof. We proceed by induction on $k+\ell$, with the base case $\min\{k,\ell\} = 2$ being trivial. For the inductive step, the key claim is that the following inequality holds:

$$r(k,\ell) \leqslant r(k-1,\ell) + r(k,\ell-1). \tag{2.1}$$

To prove (2.1), fix a red/blue coloring of $E(K_N)$, where $N = r(k-1,\ell) + r(k,\ell-1)$, and fix some vertex $v \in V(K_N)$. Suppose for the moment that v is incident to at least $r(k-1,\ell)$ red edges, and let R denote the set of endpoints of these red edges. By definition, as $|R| \geqslant r(k-1,\ell)$, we know that R contains a red K_{k-1} or a blue K_ℓ. In the latter case we have found a blue K_ℓ (so we are done), and in the former case we can add v to this red K_{k-1} to obtain a red K_k (and we are again done).

So we may assume that v is incident to fewer than $r(k-1,\ell)$ red edges. By the exact same argument, just interchanging the roles of the colors, we may assume that v is incident to fewer than $r(k,\ell-1)$ blue edges. But then the total number of edges incident to v is at most

$$(r(k-1,\ell)-1) + (r(k,\ell-1)-1) = N-2,$$

which is impossible, as v is adjacent to all $N-1$ other vertices. This is a contradiction, proving (2.1).