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PROBABILISTIC INTERPRETATION OF QUANTUM FIELD THEORIES
[*after* Guillarmou, Kupiainen, Rhodes, Vargas...]

by **Martin Hairer**

1. Introduction

In this note we provide a gentle introduction to a small selection of the concepts and intuition behind the recent breakthrough results by Guillarmou, Kupiainen, Rhodes, and Vargas (2021, 2024), and coauthors[1] (see in particular the review article by Guillarmou, Kupiainen, and Rhodes, 2024) on the mathematically rigorous construction and analysis of a completely integrable non-trivial 2D conformal field theory, namely the so-called Liouville theory first introduced by Belavin, Polyakov, and Zamolodchikov (1984), Knizhnik, Polyakov, and Zamolodchikov (1988), Polyakov (1981), and Polyakov (2008). We will spend some time on trying to understand "0-dimensional QFT", i.e. simply a single quantum mechanical particle, but from the perspective of rigorous path integration. This will naturally lead us to a form of Segal's axioms, which we then generalise to the two-dimensional conformal case.

We will then discuss in some detail how the Gaussian free field, reweighted by some local and coercive potential V fits into this framework and satisfies Segal's axioms. In the last section, we will finally introduce the case of Liouville theory. Besides the functorial properties encoded in Segal's axions, this theory also exhibits a form of conformal invariance which we discuss. In particular, we will motivate the definition of the theory's central charge, as well as the Seiberg bounds on the weights of its insertions. Due to a lack of both time and space, we'll leave out most of the recent advances in this area, in particular the proof of the DOZZ formula and the proof of the conformal bootstrap. The hope is rather that after going through these notes the reader is equipped with some of the background material required to read the original articles.

Most of the material of this note is based on Guillarmou, Kupiainen, and Rhodes (2024) and Guillarmou, Kupiainen, Rhodes, and Vargas (2021), with significant inspiration from Pickrell (2008).

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[1] Baverez, Guillarmou, Kupiainen, Rhodes, and Vargas, 2024; David, Kupiainen, Rhodes, and Vargas, 2016; Kupiainen, Rhodes, and Vargas, 2018, 2019, 2020.

2. The 0-dimensional case

Before we turn to quantum field theories, let us recall the path integral formulation of a classical one-particle quantum system. This can be interpreted as a "0-dimensional QFT" and we will take this as a starting point to "guess" what a higher-dimensional QFT should look like. Throughout this article, we will only consider spin-less particles, which are then necessarily bosons. Given a potential function $V\colon \mathbf{R}^d \to \mathbf{R}$ which we are going to assume smooth and bounded from below, the quantum mechanical Hamiltonian H describing the motion of a particle in the potential V is the operator

$$H = -\tfrac{1}{2}\Delta + V\;, \tag{1}$$

where Δ denotes the usual Laplacian on $\mathbf{R}^d$. We can realise H as a selfadjoint operator on $\mathscr{H} = L^2(\mathbf{R}^d)$ by noting that the quadratic form

$$B(\Phi,\Psi) = \int \big(\langle \nabla\Phi(x), \nabla\Psi(x)\rangle + V(x)\Phi(x)\Psi(x)\big)\,dx\;,$$

defined on $\mathscr{C}_0^\infty \subset \mathscr{H}$ is symmetric, positive and closable. Writing $\mathscr{D}(B)$ for the domain of its closure, it is a classical fact (Friedrichs, 1934; Kato, 1995) that the operator H such that

$$\mathscr{D}(H) = \{\Phi \in \mathscr{D}(B)\,|\, \exists \hat\Phi \in \mathscr{H}\ \forall \Psi \in \mathscr{D}(B)\,:\, \langle \hat\Phi, \Psi\rangle = B(\Phi,\Psi)\} \tag{2}$$

and $H\Phi = \hat\Phi$ (with $\hat\Phi$ given as in (2), which is unique by the density of $\mathscr{D}(B)$ in $\mathscr{H}$ and Riesz's representation theorem) is indeed selfadjoint and agrees with (1) on $\mathscr{C}_0^\infty$.

Given such a Hamiltonian (which will in general be a selfadjoint operator that is bounded from below), a *state* ψ for the corresponding quantum-mechanical system is a ray[(2)] in the complexification of the Hilbert space $\mathscr{H}$. The evolution of such a state is then given by the solution to the Schrödinger equation, namely

$$\partial_t \psi = -iH\psi\;,$$

where we identify ψ with one of its representatives in $\mathscr{H}$ (by linearity it doesn't matter which one). This is of course nothing but the strongly continuous group of unitary operators generated by the anti self-adjoint operator $-iH$. Conversely, given such a group, one can recover H uniquely (but it need not be bounded from below).

2.1. Path integral representation

On the other hand, given H as above, we can consider the *semigroup* P_t^V generated by $-H$ and, conversely, any strongly continuous semigroup of selfadjoint operators on

(2) namely a linear subspace of (complex) dimension one

$\mathscr{H}$ is generated by a selfadjoint operator that is bounded from below. Therefore, in order to identify H and therefore to "construct" a quantum field theory, it is sufficient to construct the corresponding "heat semigroup" P_t^V (since this is precisely what it is when $V \equiv 0$). As a consequence of the Feynman–Kac formula (KAC, 1949), one has the following stochastic representation of P_t^V:

$$\big(P_t^V F\big)(x) = \mathbf{E}_x\Big(F(\Phi_t)\exp\Big(-\int_0^t V(\Phi_s)\,ds\Big)\Big)\;. \tag{3}$$

Here, under the expectation $\mathbf{E}_x$, Φ is a standard Brownian motion starting from the location x at time 0. The reason why we use this strange notation (as opposed to B or W) is that Φ will play the role of a field later on.

A very fruitful idea that leads to the "correct" intuition is to formally rewrite (3) as integral against the non-existing "Lebesgue measure" on the space of functions. For this, recall that if μ is a Gaussian measure on $\mathbf{R}^N$ with covariance matrix C, then it has a density with respect to Lebesgue measure given by

$$\mu(dx) = \frac{1}{\sqrt{\det(2\pi C)}}\exp\big(-\tfrac{1}{2}\langle x, C^{-1}x\rangle\big)\,dx\;. \tag{4}$$

In the case of Brownian motion, its covariance operator C is the integral operator on $L^2([0,T])$ (say) with kernel given by $C(s,t) = s \wedge t$. If $\Phi\colon [0,T] \to \mathbf{R}$ is a smooth function with $\Phi(0) = 0$ and $\dot\Phi(T) = 0$, then an integration by parts on the first term shows that

$$\begin{aligned}\big(C\ddot\Phi\big)(t) &= \int_0^t s\ddot\Phi(s)\,ds + t\int_t^T \ddot\Phi(s)\,ds \\ &= t\dot\Phi(t) - \int_0^t \dot\Phi(s)\,ds - t\dot\Phi(t) = -\Phi(t)\;.\end{aligned}$$

This shows that C^{-1} is nothing but the operator $-\partial_t^2$ with the abovementioned boundary conditions. In view of (4), it is then very tempting to rewrite (3) as

$$\big(P_t^V F\big)(x) = Z^{-1}\int F(\Phi_t)\delta_x(\Phi_0)\exp\Big(-\int_0^t \big(\tfrac{1}{2}|\dot\Phi_s|^2 + V(\Phi_s)\big)\,ds\Big)\,d\Phi\;, \tag{5}$$

for some normalisation constant Z. This is of course completely nonsensical (for starters the constant Z involves a factor $(2\pi)^\infty$ and the determinant of the unbounded operator $-\partial_t^2$), but it is nevertheless a powerful guide for our intuition of what a QFT should be.

In particular, the semigroup property of P_t^V (which we recall is crucial in order to extract from it the Hamiltonian operator H) now appears naturally as a consequence of the fact that the expression appearing inside the exponential is the integral of a local expression of the field Φ. At the mathematically rigorous level, we recall that given $x, y \in \mathbf{R}^d$, a Brownian motion Φ starting at $\Phi_0 = x$ and conditioned to have $\Phi_t = y$ can be decomposed as

$$\Phi_s = \tilde{\Phi}_s + \frac{sy + (t-s)x}{t} , \tag{6}$$

where $\tilde{\Phi}$ is a Brownian bridge. We deduce from this and the fact that Φ_t is a Gaussian random variable with mean x and variance t, that P_t^V is an integral operator with kernel given by

$$P_t^V(x,y) = P_t(x,y)\, \mathbf{E} \exp\Big(- \int_0^t V\Big(\tilde{\Phi}_s + \frac{sy + (t-s)x}{t}\Big)\, ds \Big) , \tag{7}$$

where P_t denotes the usual heat kernel. One nice feature of this representation is that, since we know that the operator P_t with kernel $P_t(x,y) \propto \exp(-\frac{|x-y|^2}{2t})$ is selfadjoint, we immediately see from (7) that P_t^V is also selfadjoint since the Brownian bridge measure is invariant under the change of variables $s \mapsto t - s$ which exchanges the roles of x and y.

2.2. Half-densities

In order to study quantum field theories, one would like to generalise constructions of the type (7) to infinite-dimensional situations. This however makes it somewhat unclear how expressions like (7) in which x and y play symmetric roles can be extended to such a situation. Indeed, one could "naïvely" think that the infinite-dimensional analogue of the "heat kernel" would be the Markov transition kernel of some infinite-dimensional Markov process (the analogue of the Brownian motion in the previous discussion). Since however there isn't any analogue of Lebesgue measure in infinite dimensions, a Markov kernel cannot be represented by a function of two variables there. Instead, it is naturally a function in its first argument, but a measure in its second argument, thus breaking the nice symmetry between x and y.

One solution to this problem is the use of so-called half-densities. These are based on the simple observation that, given two positive Radon measures μ_1, μ_2 on a (Polish) space $\mathscr{X}$, we can canonically define a measure $\sqrt{\mu_1\mu_2}$ by

$$\sqrt{\mu_1\mu_2}(A) = \int_A \sqrt{\frac{d\mu_1}{d\nu}(x)\frac{d\mu_2}{d\nu}(x)}\, \nu(dx) ,$$

where ν is any positive Radon measure such that $\mu_i \ll \nu$ (for example $\nu = \mu_1 + \mu_2$)[3]. It is not difficult to prove (and already apparent from the notation) that this expression is indeed independent of the choice of reference measure ν.

(3) Here and below we write $\mu \ll \nu$ to mean that μ is absolutely continuous with respect to ν.

Given a measure class $[\nu]$ on $\mathscr{X}$, we then have a Hilbert space $\mathscr{H}_{[\nu]}$ which is formally nothing but $L^2(\mathscr{X},\nu)$, but we think of its elements as expressions of the type $f\sqrt{\nu}$ with $f \in L^2(\mathscr{X},\nu)$, endowed with the scalar product

$$\langle f\sqrt{\nu}, \tilde f\sqrt{\tilde\nu}\rangle = \int_{\mathscr{X}} f(x)\tilde f(x)\,\sqrt{\nu\tilde\nu}(dx)\;,$$

as well as the natural equivalence relation postulating that $f_1\sqrt{\nu_1} \sim f_2\sqrt{\nu_2}$ if and only if there exists a measure μ with $\nu_i \ll \mu$ such that $f_1\sqrt{\frac{d\nu_1}{d\mu}} = f_2\sqrt{\frac{d\nu_2}{d\mu}}$. In this way, $\mathscr{H}_{[\nu]}$ is defined in a canonical way that only depends on the measure class $[\nu]$ and not on its particular choice of representative ν.

Remark 2.1. It is not difficult to see that these spaces have the same tensorial property as the usual L^2 spaces, namely, given measure spaces $(\mathscr{X},\nu)$ and $(\mathscr{Y},\mu)$ as above, we have $\mathscr{H}_{[\mu\otimes\nu]} \simeq \mathscr{H}_{[\mu]} \otimes \mathscr{H}_{[\nu]}$, with $\otimes$ denoting the tensor product of Hilbert spaces (which is again a Hilbert space). Here and below, we use the symbol $\simeq$ to denote that two objects are not just isomorphic but *canonically* isomorphic, so can be identified for all intents and purposes.

We now remark that (7) can be written in a natural way as a half-density in the following way. Consider the σ-finite measure $\hat{\mathbf{P}}_t$ on $\mathscr{C}([0,2t],\mathbf{R}^d)$ given by $\hat{\mathbf{P}}_t(d\Phi) = \int_{\mathbf{R}^d}\big(\tau_*^{(c)}\mathbf{P}_{2t}^{(0)}\big)(d\Phi)\,dc$, where $\mathbf{P}_{2t}^{(0)}$ denotes the law of a Brownian bridge on $[0,2t]$ (which is therefore a probability measure on $\mathscr{C}([0,2t],\mathbf{R}^d)$) and $(\tau^{(c)}\Phi)(s) = \Phi(s)+c$. We can then consider the measure $\hat{\mathbf{P}}_t^V$ given by

$$\hat{\mathbf{P}}_t^V(d\Phi) = \exp\Big(-\int_0^{2t} V\big(\Phi_s\big)\,ds\Big)\,\hat{\mathbf{P}}_t(d\Phi)\;. \tag{8}$$

If V grows sufficiently fast at infinity (any strictly positive power of its argument will do), then one can show that the measure $\hat{\mathbf{P}}_t^V$ is finite. Consider furthermore the map $\pi\colon \mathscr{C}([0,2t],\mathbf{R}^d) \to \mathbf{R}^d\times\mathbf{R}^d$ given by

$$\pi\Phi = (\Phi_0,\Phi_t)\;.$$

We then claim the following.

Proposition 2.2. *With P_t^V as in (7) one has $P_t^V\sqrt{dx\,dy} = \sqrt{(4\pi t)^{-d/2}\pi_*\hat{\mathbf{P}}_t^V}$.*

Proof. given $x,y\in\mathbf{R}^d$, write $\Phi_{x,y}\colon [0,2t]\to\mathbf{R}^d$ for the function that is affine on $[0,t]$ and $[t,2t]$ and such that $\Phi_{x,y}(0) = \Phi_{x,y}(2t) = x$ and $\Phi_{x,y}(t) = y$. We then consider the bijection

$$\begin{aligned}\Xi\colon \mathscr{C}_0([0,t],\mathbf{R}^d)\times\mathbf{R}^d\times\mathbf{R}^d\times\mathscr{C}_0([0,t],\mathbf{R}^d) &\to \mathscr{C}([0,2t],\mathbf{R}^d)\\ (\Phi,x,y,\hat\Phi) &\mapsto \Phi_{x,y} + (\Phi\sqcup\hat\Phi)\;,\end{aligned}$$