Séminaire BOURBAKI Février 2025
76e année, 2024–2025, no 1234, p. 221 à 256
doi: 10.24033/ast.1259

SPECTRAL THEORY OF NONLOCAL OPERATORS AND INFINITE DIMENSIONAL INTEGRABLE SYSTEMS
[*after* P. Gérard, S. Grellier, T. Kappeler and P. Topalov]

by **Nikolay Tzvetkov**

1. Introduction

If one wishes to solve the equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are complex numbers then it is useful to observe that $y = x + \frac{b}{2a}$ solves $y^2 = \alpha$, where $\alpha = \frac{b^2-4ac}{4a^2}$. In other words, in the variable y the equation has a simpler form. One can proceed similarly for equations of higher degree but, as it is well known, the situation becomes more involved.

One can use a similar strategy for solving constant coefficients linear ordinary differential equations (ODE). Indeed, let A be a $n \times n$ complex matrix and consider the ODE $\dot{x}(t) = Ax(t)$, where the vector $x(t) \in \mathbb{C}^n$ is unknown. Suppose that A is diagonalizable and that the matrix T is such that $TAT^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then the components of the vector $(y_1(t), \dots, y_n(t)) = Tx(t)$ solve the equations $\dot{y}_j(t) = \lambda_j y_j(t)$, $1 \leqslant j \leqslant n$, the solutions of which are given by $y_j(t) = e^{\lambda_j t} y_j(0)$. Therefore, again in the new variables $(y_1(t), \dots, y_n(t))$ the equation we aim to solve takes a simpler form. One can perform a similar reasoning if A is not diagonalizable by using the Jordan normal form reduction.

Let us now apply the same strategy to the class of Hamiltonian ODE which are closely related to the main matter of this text. Consider therefore the ODE

$$\dot{q}(t) = \frac{\partial H}{\partial p}(q(t), p(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)), \tag{1}$$

where $q(t) \in \mathbb{R}^n$ and $p(t) \in \mathbb{R}^n$ are the unknown. The equation (1) can be written as $(\dot{q}(t), \dot{p}(t)) = J\nabla_{q,p} H(q,p)$, where J is the anti-symmetric operator on $\mathbb{R}^{2n}$ defined by $J(q,p) = (p, -q)$. The operator J may be replaced by other anti-symmetric maps and we still get Hamiltonian ODE. The function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is called the Hamiltonian of the system of ODE (1). Recall that the Newton law $\ddot{x}(t) = \nabla V(x(t))$ can be written under the form (1), for $(q(t), p(t)) = (x(t), \dot{x}(t))$ with $H(q,p) = \frac{p^2}{2} - V(q)$.

As a direct consequence of (1), we obtain that $H(q(t), p(t))$ is a conserved quantity under the evolution (a conservation law). In the case $n = 1$ this conservation law alone suffices to integrate (1) by the separation of variables method for scalar ODE. For $n > 1$, the situation becomes more involved and in order to reduce (1) to a simpler system new conservation laws are needed. Fortunately, in many interesting situations such conservation laws exist. Let F_1 and F_2 be two conservation laws of (1). We say that F_1 and F_2 are in involution if $(J\nabla_{q,p}F_1(q,p), \nabla_{q,p}F_2(q,p)) = 0$, where $(\cdot,\cdot)$ stays for the $\mathbb{R}^{2n}$ scalar product. Suppose that (1) has n conservation laws $F_1, \cdots, F_n$ which are pairwise in involution and suppose that $(\nabla_{q,p}F_1, \cdots, \nabla_{q,p}F_n)$ are linearly independent on a dense open set. A constant solution of (1) is called an elliptic equilibrium if the spectrum of the linearization about it is purely imaginary. Thanks to Ito (1989), Rüssmann (1964), and Vey (1978) it is known that if an elliptic equilibrium satisfies a non resonant condition on the spectrum of the linearization then near this equilibrium one can introduce coordinates $(x,y) = (x(q,p), y(q,p))$ such that in the coordinates (x,y) the equation (1) is reduced to

$$\dot{x}(t) = \frac{\partial \mathscr{H}}{\partial y}(x(t),y(t)), \quad \dot{y}(t) = -\frac{\partial \mathscr{H}}{\partial x}(x(t),y(t)), \tag{2}$$

where the new Hamiltonian $\mathscr{H}(x,y) = \mathscr{H}(x_1,\cdots,x_n,y_1,\cdots,y_n)$ is given by

$$\mathscr{H}(x,y) = G(x_1^2+y_1^2,\cdots,x_n^2+y_n^2),$$

where $G\colon \mathbb{R}^n \to \mathbb{R}$ depends only on n variables. The coordinates (x,y) are called local Birkhoff coordinates[(1)]. By setting $z_j(t) = x_j(t) + iy_j(t)$, we observe that the solution of (2) is given by

$$z_j(t) = \exp\big(-2it\partial_j G(|z_1(0)|^2,\cdots,|z_n(0)|^2)\big)\, z_j(0), \quad 1 \leqslant j \leqslant n, \tag{3}$$

where $\partial_j G$ denotes the partial derivative of G with respect to the j^{th} variable. Again, we reduced the initial problem of solving (1) to the much simpler problem of solving (2). Usually, we apply Ito (1989), Rüssmann (1964), and Vey (1978) to make such a reduction locally around a point and therefore it is a local theorem. If we are lucky enough, these coordinates may work globally as well. Looking at (3) we observe that the motion is taking place on an $(n-k)$-dimensional torus where k is the number of vanishing $z_j(0)$. This flexibility of the dimension of the invariant tori is related to the assumption that $(\nabla_{q,p}F_1,\cdots,\nabla_{q,p}F_n)$ are linearly independent only on a dense open set. Recall that in the Liouville–Arnold theorem such an assumption is made everywhere and therefore the invariant tori are of maximal dimension.

(1) One may wish to state the existence of Birkhoff coordinates in terms of the existence of a canonical map on a symplectic manifold.

In the 19th century there were many studies in which, in the spirit if the previous paragraph, conservation laws were used to find good coordinates for Hamiltonian ODE. A famous work is the one by Jacobi dealing with the geodesic flow on the surface of a three dimensional ellipsoid. Another well known work is by Liouville who proved the local part of what is nowadays known as the Liouville–Arnold theorem.

Using conservation laws for solving Hamiltonian partial differential equations (PDE) is a much more recent subject. Intuitively, one may see a Hamiltonian PDE as a Hamiltonian ODE with infinite degrees of freedom (the system (1) with $n = \infty$). In the case of finitely many degrees of freedom, in order to start to look for suitable good coordinates one needs at least half of the degrees of freedom number of independent (in a suitable sense) conservation laws. Therefore in the case of a PDE one would need infinitely many independent conservation laws in order to start hoping to find good coordinates. Such a property may seem too optimistic for being true. However, in Gardner, Greene, Kruskal, and Miura (1967), using experimental methods it was discovered that the Korteweg-de Vries (KdV) equation has infinitely many independent conserved quantities[(2)]. Soon after Lax (1968) discovered a systematic way for deriving infinitely many conservation laws for equations having a particular structure which will be explained below. In the years which followed these developments, global Birkhoff coordinated in the context of the KdV equation were introduced (see the book Kappeler and Pöschel, 2003 and the many remarkable references therein). Namely, the KdV equation was written globally in the form (2) (with $n = \infty$).

Let us next describe the Lax method in the context of the KdV equation, posed on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This setting is in a sense the closest to the finite dimensional situation described in (1). For this reason in the whole text we will remain in this setting of periodic in space solutions. The KdV equation, posed on $\mathbb{T}$ reads

$$\partial_t u = \partial_x(-\partial_x^2 u + 3u^2), \tag{4}$$

where $u\colon \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ is the unknown with a prescribed value at $t = 0$ as a function (or distribution) in a suitable analytic framework. One may write (4) in the Hamiltonian form $\partial_t u = J\nabla H(u)$, where $J = \partial_x$ (an anti-symmetric map with respect to the L^2 scalar product) and $H(u) = \frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2 + \int_{\mathbb{T}} u^3$. This Hamiltonian structure alone does not give any hint on how to look for other conservation laws than the Hamiltonian $H(u)$. The extraordinary observation of Lax (1968) is that if $u(t)$ solves (4) then

$$\frac{d}{dt} L_{u(t)} = [B_{u(t)}, L_{u(t)}], \tag{5}$$

(2) The KdV equation is a partial differential equation obtained as an asymptotic model derived from the water waves system for the propagation of long, one directional small amplitude surface waves in a shallow water (see e.g. Lannes, 2013).

where the linear maps L_u and B_u are defined by

$$L_u(v) = -\partial_x^2 v + uv, \quad B_u(v) = -4\partial_x^3 v + 3\partial_x(uv) + 3u\partial_x v.$$

The pair (L_u, B_u) is called a Lax pair and (5) is called a Lax pair formulation of the KdV equation (4). Clearly the operator L_u is symmetric and the operator B_u is anti-symmetric with respect to the (real) L^2 scalar product. As we shall see in the next paragraph, a key consequence of the above formulation is that the spectrum of $L_{u(t)}$ is independent of t. In other words for every t the solution of (4) belongs to the iso-spectral set of $L_{u(0)}$ and every function of the spectrum of $L_{u(0)}$ is a conservation law of (4). This is of course a remarkable fact.

Let $U(t)$ be the solution of the operator valued linear ODE

$$\frac{d}{dt}U(t) = B_{u(t)}U(t), \quad U(0) = \mathrm{Id}. \tag{6}$$

Since $B(t)$ is anti-symmetric $((B(t))^\star = -B(t))$, we have that

$$(U(t))^\star = (U(t))^{-1}. \tag{7}$$

Differentiating in t the identity $\mathrm{Id} = (U(t))^{-1} \circ U(t)$, and using (6), we obtain that

$$\frac{d}{dt}(U(t))^{-1} = -(U(t))^{-1} \circ B_{u(t)}. \tag{8}$$

Therefore, using the Leibniz rule, (6), (8) and (5), we get

$$\frac{d}{dt}\big((U(t))^{-1} \circ L_{u(t)} \circ U(t)\big) = (U(t))^{-1} \circ \big(\frac{d}{dt}L_{u(t)} + [L_{u(t)}, B_{u(t)}]\big) \circ U(t) = 0.$$

Coming back to (7), we get the key relation

$$L_{u(t)} = U(t) \circ L_{u(0)} \circ (U(t))^\star.$$

The spectral theory of L_u can be analyzed via the Sturm–Liouville theory which is a well-established branch in the theory of second order linear ODE. Thanks to this ODE theory and the Lax pair formulation, one may define the Birkhoff coordinates for the KdV equation, see Kappeler and Pöschel (2003) for a textbook presentation. One remarkable consequence of the Birkhoff coordinates is that the solutions of (4) are almost periodic in time which is a deep insight in the long time dynamics.

Let us next turn to the Benjamin–Ono (BO) equation. This equation was derived as a model for long, one directional, small amplitude internal waves (see e.g. Klein and Saut, 2021). The BO equation, posed on $\mathbb{T}$ reads

$$\partial_t u = \partial_x(|D|u - u^2), \tag{9}$$

where $u\colon \mathbb{R}\times\mathbb{T}\to\mathbb{R}$ is the unknown. The operator $|D|$ is defined by $H\partial_x$, where H is the Hilbert transform on $\mathbb{T}$. In other words, one defines $|D|$ via the Fourier transform by $\widehat{|D|u}(n) = |n|\hat{u}(n)$ for every integer n which shows that $|D|$ is a positive operator, after invoking the Plancherel identity. One may write (9) in a Hamiltonian form similarly to (4). One can also observe that (4) and (9) have a similar structure, the main difference is that the second order positive operator $-\partial_x^2$ is replaced by the first order positive (necessarily non local) operator $|D|$. It is therefore probably not so surprising that the solutions of (9) can also satisfy a Lax identity of type (5) but this time with nonlocal operators L_u and B_u. Let us introduce these operators precisely. We denote by $L^2_+(\mathbb{T})$ the Hardy space of $L^2(\mathbb{T})$ functions f such that $\hat{f}(n)=0$ for $n<0$. Such functions can be written as $\sum_{n\geqslant 0} e^{inx}\hat{f}(n)$ and can be seen as the boundary values of the holomorphic functions on the unit disc $\{z\in\mathbb{C}\colon |z|<1\}$ defined by $\sum_{n\geqslant 0}\hat{f}(n)\,z^n$. Therefore we will naturally identify a function in $L^2_+(\mathbb{T})$ and its holomorphic extension. Typically, if u solves (4) or (9) with a square integrable initial datum at $t=0$ then $\Pi(u)$ belongs to $L^2_+(\mathbb{T})$, where the projector Π is defined by

$$\Pi(u)(x) = \sum_{n\geqslant 0} e^{inx}\,\hat{u}(n)\,.$$

Moreover, the knowledge of $\Pi(u)$ implies the knowledge of u because the solutions of (4) or (9) are real valued. The Toeplitz operator $T_b\colon L^2_+(\mathbb{T})\to L^2_+(\mathbb{T})$ associated with a function $b\in L^\infty(\mathbb{T})$ is defined by $T_b(u)=\Pi(bu)$. The Sobolev spaces $H^s(\mathbb{T})$ are defined by the norm

$$\|u\|^2_{H^s} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2s}\,|\hat{u}(n)|^2\,.$$

We denote by $H^s_r(\mathbb{T})$ the closed subspace of real valued elements of $H^s(\mathbb{T})$. Next, for $u\in H^s_r(\mathbb{T})$, $s\geqslant 0$, we denote by $L_u\colon L^2_+(\mathbb{T})\to L^2_+(\mathbb{T})$ the operator

$$L_u(v) = |D|v - T_u(v). \tag{10}$$

The operator L_u is self-adjoint on $L^2_+(\mathbb{T})$ with domain $L^2_+(\mathbb{T})\cap H^1(\mathbb{T})$. For $u\in H^s_r(\mathbb{T})$, $s\geqslant 0$, we denote by B_u the anti-symmetric operator on $L^2_+(\mathbb{T})$ defined by

$$B_u = i(T_{|D|u} - T_u^2). \tag{11}$$

We have that B_u is bounded for $s\geqslant 2$, thanks to basic properties of the projector Π and a Sobolev embedding. In strong analogy with KdV, it was observed in Fokas and Ablowitz (1983) and Nakamura (1979) that if $u(t,x)$ is a C^∞ solution of the BO equation (9) then it satisfies

$$\frac{d}{dt}L_{u(t)} = [B_{u(t)}, L_{u(t)}],$$