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SOLITON RESOLUTION FOR ENERGY CRITICAL WAVE TYPE EQUATIONS
[*after* **Duyckaerts–Kenig–Merle and Jendrej–Lawrie**]

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Introduction

Dispersive equations are evolution partial differential equations for which plane waves have speed which depends on their frequency (at a linear level). As a consequence, the bulk of a solution to the linear part of the equation tends to split spatially as time grows, or disperse; this explains the terminology. The Schrödinger equation, the Korteweg–de Vries equation, or the wave equation are among the most prominent examples of dispersive equations.

For the nonlinear version of these equations, one distinguishes often whether the nonlinearity in some sense “helps” the dispersion (one says that the nonlinearity or the equation is defocusing) or to the contrary, tends to make solution concentrate (one speaks of a focusing nonlinearity). For focusing nonlinearity, there often exists special, non-trivial non-linear objects: they can be stationary, standing wave or travelling wave solution and are called soliton. Their key feature is that their shape remains the same through the evolution. Solitons realise a kind of balance between the focusing nonlinearity and the dispersive part of the equation.

Somewhat surprisingly, it was observed numerically as soon as ZABUSKY and KRUSKAL (1965) that, from an a priori unspecific initial data, the solution of the Korteweg-de Vries equation would split into a sum of solitons (which are in this case travelling waves), as time goes large. The numerical simulation was done in a periodic setting, so that solitons would collide (interact) repetitively with one another, but would come out of this interaction without change of shape: this is refered to as *elastic* collision. This is surprising simplification of the dynamics, where for large times, solutions can be described using only solitons. It led to the so-called soliton resolution conjecture: it is somewhat vague, and asserts that this simplification phenomenon occurs for generic data in most non-linear dispersive models.

In the 1970s, the theory of integrable systems was developed, in part to study this conjecture. It led to the inverse scattering method, a very powerful tool which gave, among other stricking results, a proof of the soliton resolution conjecture for some equations, and most notably, for the Korteweg-de Vries equation (there are still some open questions remaining regarding the long time dynamics, though). Nevertheless, for non integrable models, the question remained widely open.

In this report, we will be interested in a series of results, over the last fifteen years, related to the soliton resolution for energy critical wave type equations. The first one is the focusing energy critical non linear wave equation (with pure power nonlinearity), and the second one is the wave map system; they are not integrable. The most elaborate statements are stated (and proved) for radially symmetric solutions (for the former) or for equivariant solutions (for the latter); however, some of intermediate results were also obtained for general data, and we will emphasise when it holds.

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1. Two wave type partial differential equations

1.1. The energy critical wave equation

The energy critical wave equation writes

$$\partial_{tt} u - \Delta u - |u|^{\frac{4}{d-2}} u = 0, \tag{NLW}$$

where here $d \geqslant 3$ is the underlying spatial dimension, $t \in \mathbb{R}$ represents time, $x \in \mathbb{R}^d$ is the space variable, and $u(t,x) \in \mathbb{R}$. We say that it is radial if u depends on x only through the radial coordinate $r = |x| \in [0, \infty)$, in which case the Laplacian writes $\Delta := \partial_r^2 + (d-1) r^{-1} \partial_r$.

It is convenient to recast (NLW) as a first order Hamiltonian system. To this end, we will write pairs of functions using boldface, $\boldsymbol{v} = (v, \dot v)$, noting that the notation $\dot v$ does not necessarily refer to the time derivative of v but just to the second component of the vector $\boldsymbol{v}$. Equation (NLW) admits a conserved energy: given a function $\boldsymbol{v} = (v, \dot v)$ depending on space, denote

$$E(\boldsymbol{v}) := \int_0^\infty \Big[\frac{1}{2} \dot v(x)^2 + \frac{1}{2} |\nabla v(x)|^2 - \frac{d-2}{2d} |v(x)|^{\frac{2d}{d-2}} \Big] dx. \tag{1}$$

Then formally, if u is a solution of (NLW) defined on a time interval $I \ni 0$, and letting $\boldsymbol{u} = (u, \partial_t u)$ there hold

$$\forall t \in I, \quad E(\boldsymbol{u}(t)) = E(\boldsymbol{u}(0)). \tag{2}$$

We now see that the Cauchy problem for (NLW) is equivalent to

$$\partial_t \boldsymbol{u}(t) = J \circ \nabla E(\boldsymbol{u}(t)), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3}$$

where J is a skew-symmetric matrix and ∇E is the formal gradient of E:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla E(\boldsymbol{v}) = \begin{pmatrix} -\Delta v - |v|^{\frac{4}{d-2}} u \\ \dot{v} \end{pmatrix}. \tag{4}$$

The linearization of (NLW) around the zero solution is the free scalar wave equation,

$$\partial_{tt} v - \Delta v = 0. \tag{5}$$

We will often denote $S_L(t)\boldsymbol{v}_0 = (v(t), \partial_t v(t))$ for the linear solution to (5) with initial data $\boldsymbol{v}_0 = (v_0, \dot{v}_0)$ at time $t = 0$: the free wave propagator $S_L(t)$ writes explicitely

$$S_L(t)\boldsymbol{v}_0 = \left(\cos(t|\nabla|)v_0 + \frac{\sin(t|\nabla|)}{|\nabla|}\dot{v}_0, -|\nabla|\sin(t|\nabla|)v_0 + \cos(t|\nabla|)\dot{v}_0 \right).$$

Solutions to (NLW) are invariant under the scaling

$$\boldsymbol{u}(t,x) \mapsto \boldsymbol{u}_\lambda(t,x) := \left(\lambda^{-\frac{d-2}{2}} u(t/\lambda, x/\lambda), \lambda^{-\frac{d}{2}} \partial_t u(t/\lambda, x/\lambda) \right), \quad \text{where} \quad \lambda > 0, \tag{6}$$

and (NLW) is called *energy critical* because, where defined,

$$E(\boldsymbol{u}(t)) = E(\boldsymbol{u}_\lambda(t)).$$

This scaling consideration makes that it would be suitable that the Cauchy problem be solved in the energy space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, which we will simply denote $\dot{H}^1 \times L^2$ for short from now on. This was done first by Ginibre and Velo (1989) and revisited by Bulut, Czubak, Li, Pavlović, and Zhang (2013) and Kenig and Merle (2008). Let us give a precise statement. To this end, we introduce the Strichartz type spaces

$$S(I) := L^{\frac{2(d+1)}{d-2}}(I \times \mathbb{R}^d),$$
$$W(I) := L^{\frac{2(d+1)}{d-1}}\left(I; \dot{B}^{\frac{1}{2}}_{\frac{2(d+1)}{d-1},2}(\mathbb{R}^d)\right),$$

where $I \subset \mathbb{R}$ is a time interval. Here the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined for $0 < s < 1$, $1 \leqslant p, q < +\infty$ as

$$\|v\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{js} \|P_j u\|^q_{L^p} \right)^{1/q},$$

where $(P_j)_{j\in\mathbb{Z}}$ are Littlewood–Paley projections⁽¹⁾. We refer to the book by Bahouri, Chemin, and Danchin (2011) for properties and details on homogeneous Besov

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(1) One can think of P_j as localizing to frequency $\simeq 2^j$: for example $\widehat{P_j v}(\xi) = \chi(|\xi|/2^j)\hat{v}(\xi)$, where $\chi : (0, +\infty) \to [0,1]$ is a smooth cut-off function with support in $[3/4, 7/3]$ and strictly positive on $[1,2]$.

spaces: we will use them only seldomly here, and only recalled them for completeness. The space $S(I)$ plays however an important role below.

We say that $\boldsymbol{u}$ is a solution to (NLW) on a time interval $I \ni 0$, with initial data $\boldsymbol{u}_0$, if

1. $\boldsymbol{u} \in \mathscr{C}(I, \dot{H}^1 \times L^2(\mathbb{R}^d))$, and $u \in S(J) \cap W(J)$ for all compact intervals $J \subset I$.

2. $\boldsymbol{u}$ is a solution of (NLW) in its Duhamel integral formulation, that is

$$\forall t \in I, \quad \boldsymbol{u}(t) = S_L(t)\boldsymbol{u}_0 + \int_0^t S_L(t-s)(0, f(u(s))ds.$$

Theorem 1.1 (Cauchy theory in $\dot{H}^1 \times L^2$, Kenig and Merle, 2008, Theorem 2.7 and Bulut, Czubak, Li, Pavlović, and Zhang, 2013, Theorem 3.3). *1) Existence and uniqueness of a maximal solution. There exists a function $\delta : [0, \infty) \to (0, \infty)$ with the following properties. Let $A > 0$ and $\boldsymbol{u}_0 = (u_0, u_1) \in \dot{H}^1 \times L^2$ with $\|\boldsymbol{u}_0\|_{\dot{H}^1 \times L^2} \leqslant A$. Let $I \ni 0$ be an open interval such that*

$$\|S_L(\cdot)\boldsymbol{u}_0\|_{S(I)} \leqslant \delta(A).$$

Then there exists a unique solution $\boldsymbol{u}(t)$ to (NLW) *in the space $\mathscr{C}^0(I, \dot{H}^1 \times L^2) \cap S(I) \cap W(I)$ with initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$.*

To each initial data $\boldsymbol{u}_0 \in \dot{H}^1 \times L^2$, we can associate a unique solution $\boldsymbol{u}$ of (NLW) *defined on a* maximal *forward interval of existence $[0, T_+)$ such that for each compact subinterval $J \subset [0, T_+)$ we have $\|u\|_{S(J)} < \infty$ and, if $T_+ < \infty$, then $\|u\|_{S([0,T_+))} = \infty$.*

We will freely write $T_+(\boldsymbol{u})$ to denote the forward maximal time of existence of a maximal solution $\boldsymbol{u}$.

2) Continuity of the flow. Let $\boldsymbol{u}_0 \in \dot{H}^1 \times L^2$ and let $\boldsymbol{u}(t) \in \dot{H}^1 \times L^2$ be the unique maximal solution to (NLW) *with initial data $\boldsymbol{u}_0$, and let $T < T_+(\boldsymbol{u})$. Then for every $\varepsilon > 0$ there exists $\eta > 0$ with the following property: for all $\boldsymbol{v}_0 \in \dot{H}^1 \times L^2$ with $\|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\dot{H}^1 \times L^2} < \eta$ we have $T_+(\boldsymbol{v}) \geqslant T$ and $\sup_{t \in [0,T]} \|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{\dot{H}^1 \times L^2} < \varepsilon$, where $\boldsymbol{v}(t)$ is the unique solution to* (NLW) *associated to $\boldsymbol{v}_0$.*

There are similar statements backward in time, with $T_-(\boldsymbol{u})$ denoting the maximal backward time of existence. In the following, all the solutions $\boldsymbol{u}$ we consider will be maximal forward; we say that $\boldsymbol{u}$ is forward global if $T_+(\boldsymbol{u}) = +\infty$, and that it is a blow up solution if $T_+(\boldsymbol{u}) < +\infty$.

As a rather direct consequence of this local well posedness result, we see that the Strichartz space $S(I)$ plays a role in the long time description, as a measure of the strength of the nonlinearity. More specifically, if $\boldsymbol{u}$ is a forward maximal solution to (NLW) (in the above sense) which satisfies

$$u \in S([0, T_+(\boldsymbol{u}))),$$

then $T_+(\boldsymbol{u}) = +\infty$ and there is linear scattering as $t \to +\infty$, that is there exists $\boldsymbol{u}^+$ in $\dot{H}^1 \times L^2$ such that

$$\|\boldsymbol{u}(t) - S_L(t)\boldsymbol{u}^+\|_{\dot{H}^1 \times L^2} \to 0 \quad \text{as } t \to +\infty. \tag{7}$$

In this case, we say that $\boldsymbol{u}$ scatters (linearly) to $\boldsymbol{u}^+$. Also notice that if $\|S_L(\cdot)\boldsymbol{u}_0\|_{S([0,+\infty))}$ is small enough, the above condition is met for the non linear solution $\boldsymbol{u}$ associated to $\boldsymbol{u}_0$, and $\boldsymbol{u}$ scatters linearly.

Conversely, the existence of wave operators holds, i.e., for any solution $S_L(\cdot)\boldsymbol{u}^+ \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ to the free linear equation, there exists a unique solution $\boldsymbol{u}$ in $\dot{H}^1 \times L^2$ to (NLW), defined for large enough times, such that (7) holds as $t \to \infty$.

Analogous statements hold for negative times.

For small dimensions $3 \leqslant d \leqslant 6$, the Besov based space $W(I)$ is not needed (as done in Kenig and Merle, 2008), but for high dimensions, the nonlinearity is no longer smooth enough, and the functional set up must be adapted: this is the input of Bulut, Czubak, Li, Pavlović, and Zhang (2013). In both cases, the above statement of Theorem 1.1, more elaborate than one could expect, is actually what is required to derive a suitable perturbation result essential in following the evolution of a profile decomposition: see Proposition 2.2 below.

At this point, we go back to the energy (1) and observe that the kinetic part of the energy and the potential (non-linear) part have opposite signs (this is related to the $+$ sign of the nonlinearity): this is caracteristic of a focusing equation.

These two effects come to a balance for the solution

$$W(x) := \Big(1 + \frac{|x|^2}{d(d-2)}\Big)^{-\frac{d-2}{2}}, \tag{8}$$

to the static (elliptic) equation

$$-\Delta W(x) = |W(x)|^{\frac{4}{d-2}} W(x). \tag{9}$$

The function W is the unique non negative (and non zero) solution to (9), up to the invariances: sign, scaling and translation. It is an extremizer for the best constant in the homogeneous Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \to L^{\frac{2d}{d-2}}(\mathbb{R}^d)$: we refer to Aubin (1978) and Talenti (1976); for this reason, W is sometimes called the Aubin–Talenti solution. Let us emphasize that the exponent in the nonlinearity in (NLW)

$$p = 1 + \frac{4}{d-2}$$

was actually chosen for the purpose of the Sobolev embedding. W plays the role of a soliton for (NLW).