Astérisque

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Deformations of algebraic varieties with G_m action

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1. Introduction

This thesis* is devoted to a study of the deformations of an isolated singularity of an algebraic variety admitting a multiplicative group action. In this section we give a summary of the techniques used and the results obtained.

(1.1) First we set up some notation and review the relevant elements of deformation theory. Details can be found in [43] and [45]. We fix once and for all an algebraically closed field k.

Let B be a local k-algebra of finite type. Let $\mathcal{C}$ be the category of local artin k-algebras, $\hat{\mathcal{C}}$ that of complete noetherian local k-algebras: hence if $A \in \hat{\mathcal{C}}$ with maximal ideal m, then $A/m^i \in \mathcal{C}$ for all i.

Definition (1.2) An infinitesimal deformation of B to $A \in \mathcal{C}$ is a cartesian diagram

$$\begin{tikzcd} B' \arrow[r] & B = B' \otimes_A k \\ A \arrow[u] \arrow[r, "res"] & k \arrow[u] \end{tikzcd}$$

where $A \longrightarrow B'$ is flat.

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<u>Definition</u> (1.3) D, the <u>local deformation functor</u> from $\mathcal{C}$ to {sets} takes $A \in \mathcal{C}$ to isomorphism classes of deformations of B to A, isomorphism being defined in the obvious way. We extend D to $\hat{\mathcal{C}}$ by taking inverse limits to get the notion of <u>formal deformation</u>.

We will usually be interested in minimal "complete" families of formal deformations, in the following sense:

<u>Definition</u> (1.4) A formal deformation $\zeta \in D(R)$, $R \in \hat{\mathcal{C}}$ is <u>versal</u> if

(i) any formal deformation $\zeta' \in D(S)$ may be deduced from ζ by a base change $R \longrightarrow S$.

(ii) the Zariski tangent space of R is minimal for (i).

R (or Spec R) is then called the <u>formal</u> <u>moduli</u> <u>space</u> of B; its tangent space (the k-vector space of first order deformations) is denoted T^1_B, or just T^1 if no confusion is possible.

Lichtenbaum-Schlessinger [28] have studied T^1 using the cotangent complex. Schlessinger's theorem [43] implies that when Spec B has an isolated singularity at its closed point, then B has a versal deformation.