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FORMS ON BERKOVICH SPACES BASED ON HARMONIC TROPICALIZATIONS

BY WALTER GUBLER, PHILIPP JELL & JOSEPH RABINOFF

ABSTRACT. — We introduce tropical skeletons for Berkovich spaces based on results of Ducros. Then we study harmonic functions on good strictly analytic spaces over a non-trivially valued non-Archimedean field. Chambert-Loir and Ducros introduced bi-graded sheaves of smooth real-valued differential forms on Berkovich spaces by pulling back Lagerberg forms with respect to tropicalization maps. We give a new approach, in which we allow pullback by more general *harmonic* tropicalizations to get a larger sheaf of differential forms with essentially the same properties but with a better cohomological behavior. A crucial ingredient is that tropical varieties arising from harmonic tropicalization maps are balanced.

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Résumé (*Fonctions classiquement psh et pluriharmoniques sur les espaces de Berkovich*). — Dans un premier temps, nous étendons la théorie des fonctions sousharmoniques sur les courbes k-analytiques strictes et lisses développée dans la thèse de Thullier, au cas des courbes analytiques possiblement singulières sur un corps non archimédien k. Les fonctions classiquement psh sont alors définies comme en géométrie complexe, à l'aide de tirés en arrière depuis des courbes analytiques (et en requérant compatibilité au changement de base). Nous établissons diverses propriétés des fonctions classiquement psh, y compris un principe du maximum local et un global. Nous en déduisons que l'espace des fonctions pluriharmoniques sur un espace de Berkovich quasi-compact est de dimension finie. Comme outil technique, nous utilisons le fait qu'un espace de Berkovich connexe est connexe par courbes analytiques.

1. Introduction

Arakelov theory is an arithmetic intersection theory on arithmetic varieties. The contributions of the Archimedean places are described analytically, while intersection theory on models over valuation rings is used for the non-Archimedean places. It is an old dream to handle all places simultaneously in an analytic manner. This would have the advantage of replacing the models, which sometimes do not exist in sufficient generality, by the underlying Berkovich analytic spaces.

It is here that tropical geometry becomes very helpful. Lagerberg [42] introduced smooth (p, q)-forms on $\mathbf{R}^n$ given in terms of coordinates by

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} \mathrm{d}' x_{i_1} \wedge \cdots \wedge \mathrm{d}' x_{i_p} \wedge \mathrm{d}'' x_{j_1} \wedge \cdots \wedge \mathrm{d}'' x_{j_q},$$

where I (resp. J) consists of $i_1 < \cdots < i_p$ (resp. $j_1 < \cdots < j_q$), and α_{IJ} are C^∞-functions on $\mathbf{R}^n$. The smooth Lagerberg forms give rise to a bigraded differential sheaf $\mathcal{A}^{\bullet,\bullet}$ of $\mathbf{R}$-algebras on $\mathbf{R}^n$ with alternating product $\wedge$ and with natural differentials d' and d'' (see 2.7). They are the analogues of the differential operators ∂ and $\overline{\partial}$ in complex analysis.

One defines smooth Lagerberg forms on a weighted rational polyhedral complex (Π, m) in $\mathbf{R}^n$ by restriction (see 2.7). This is analogous to the definition of differential forms on singular complex analytic spaces. In applications, (Π, m) will be a tropical variety. One has integrals and boundary integrals over (Π, m) satisfying the formula of Stokes.

From now on, we consider a good, d-dimensional strictly analytic Berkovich space X over a non-trivially valued non-Archimedean field K. For the basic notions used from non-Archimedean geometry, we refer to §2.3. For a compact analytic subdomain W of X, a *smooth tropicalization map* $F: W \to \mathbf{R}^n$ is given by $F(x) = (-\log|f_i(x)|)_{i=1,\ldots,n}$ for invertible analytic functions f_i on W. Chambert-Loir and Ducros [18] used smooth tropicalization maps to lift smooth Lagerberg forms to X. In this way, they obtained a bigraded differential sheaf $\mathcal{A}_{\mathrm{sm}}^{\bullet,\bullet}$ of $\mathbf{R}$-algebras on X with alternating product $\wedge$ and with natural differentials d' and d'' (see 2.8).

This opens the door to the dream mentioned above. Many results hold as expected from complex geometry. Chambert-Loir and Ducros proved that there are analogues of smooth partitions of unity, of the formula of Stokes and of the Poincaré–Lelong formula. The d''-operator induces an analogue of the Dolbeault complex, and we can introduce the tropical Dolbeault cohomology by

$$H_{\mathrm{sm}}^{p,q}(X) := \ker(\mathcal{A}_{\mathrm{sm}}^{p,q}(X) \xrightarrow{\mathrm{d}''} \mathcal{A}_{\mathrm{sm}}^{p,q+1}(X))/\mathrm{d}''(\mathcal{A}_{\mathrm{sm}}^{p,q-1}(X)). \tag{1.1}$$

It was shown in [37] that the sheaves $\mathcal{A}_{\mathrm{sm}}^{p,q}$ satisfy a local Poincaré lemma. If $X = Y^{\mathrm{an}}$ for a separated smooth scheme Y of finite type over K, then Liu [44] showed that there is an analogue of the cycle class map. However, it was realized in [39] that the cohomology groups $H_{\mathrm{sm}}^{1,1}(X)$ can be infinite dimensional and that Poincaré duality can fail for smooth projective curves X over K. The main problem can already be found in Thuillier's thesis [54] and is a result of the fact that *not all harmonic functions on curves are smooth*, as explained in Wanner's Master's thesis [56].

It was suggested in [38] that we can overcome these problems by using more general tropicalization maps based on harmonic functions. The present paper gives the foundations of such a theory. With this approach, we will get a larger bigraded differential sheaf $\mathcal{A}^{\bullet,\bullet}$ of $\mathbf{R}$-algebras on X consisting of *weakly smooth forms*. The weakly smooth forms have all the nice properties of smooth forms, but they have a better cohomological behavior.

The notation and prerequisites for this paper are explained in Section 2. We will use the smooth forms and currents introduced in [18]. Some of their definitions are recalled later as a special case of our more general construction.

1.1. Tropical skeletons and tropical multiplicities. — In Section 3, we assume that X is compact and of pure dimension d, and we consider a morphism $\varphi\colon X \to \mathbf{T}^{\mathrm{an}}$, where $\mathbf{T} = \mathbf{G}_{\mathrm{m}}^n$ is an n-dimensional split torus over K. In the case $d = n$, Ducros [19] defined the skeleton $S_\varphi(X)$ as the preimage of the canonical skeleton of $\mathbf{T}^{\mathrm{an}}$, and he showed that the skeleton has the structure of a piecewise linear space only depending on its underlying set and the analytic structure of X. Using Temkin's graded reductions, we will generalize this construction and define a *tropical skeleton* $S_\varphi(X)$ for arbitrary n. By Proposition 3.12, it is a closed subset of X consisting only of Abhyankar points, and it depends only on the smooth tropicalization map $F\colon X \to \mathbf{R}^n$ induced by φ. We will see in Remark 3.20 that the tropical skeleton $S_\varphi(X)$ is a piecewise linear space of dimension at most d closely related to the tropical variety $F(X)$. Indeed, the restriction of F induces a surjective piecewise linear map $S_\varphi(X) \to F(X)$ with finite fibers. We will show in Remark 3.19 that $S_\varphi(X)$