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ON THE BOUCKSOM–ZARISKI DECOMPOSITION FOR IRREDUCIBLE SYMPLECTIC VARIETIES AND BOUNDED NEGATIVITY

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Abstract. — Zariski decomposition plays an important role in the theory of algebraic surfaces due to many applications. For irreducible symplectic manifolds, Boucksom provided a characterization of his divisorial Zariski decomposition in terms of the Beauville–Bogomolov–Fujiki quadratic form. Different variants of singular holomorphic symplectic varieties have been extensively studied in recent years. In this note, we first show that the "Boucksom–Zariski" decomposition holds for effective divisors in the largest possible framework of varieties with symplectic singularities. On the other hand, in the case of projective surfaces, it was recently shown that there is a strict relation between the boundedness of coefficients of Zariski decompositions of pseudoeffective integral divisors and the bounded negativity conjecture. In the present

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note, we show that an analogous phenomenon can be observed in the case of projective irreducible symplectic varieties. We furthermore prove an effective analog of the bounded negativity conjecture in the smooth case. Combining these results we obtain information on the denominators of "Boucksom–Zariski" decompositions for holomorphic symplectic manifolds. From such a bound we easily deduce a result of effective birationality for big line bundles on projective holomorphic symplectic manifolds, answering a question asked by F. Charles.

Résumé (*Sur la décomposition de Boucksom-Zariski pour les variétés symplectiques irréductibles et la négativité bornée*). — La décomposition de Zariski joue un rôle important dans la théorie des surfaces algébriques en raison de ses nombreuses applications. Pour les variétés symplectiques irréductibles, Boucksom a fourni une caractérisation de sa décomposition de Zariski divisorielle en termes de la forme quadratique de Beauville-Bogomolov-Fujiki. Différentes variantes des variétés symplectiques holomorphes singulières ont été largement étudiées ces dernières années. Dans cette note, nous montrons tout d'abord que la décomposition de « Boucksom-Zariski » est valable pour les diviseurs effectifs dans le cadre le plus large possible de variétés avec singularités symplectiques. D'autre part, dans le cas des surfaces projectives, il a été récemment démontré qu'il existe une relation étroite entre la bornitude des coefficients des décompositions de Zariski des diviseurs intégraux pseudo-effectifs et la conjecture de la négativité bornée. Dans la présente note, nous montrons qu'un phénomène analogue peut être observé dans le cas des variétés symplectiques irréductibles projectives. Nous prouvons en outre un analogue effectif de la conjecture de la négativité bornée dans le cas lisse. En combinant ces résultats, nous obtenons des informations sur les dénominateurs des décompositions de Boucksom-Zariski pour les variétés symplectiques holomorphes. À partir d'une telle borne, nous déduisons facilement un résultat de birationalité effective pour les fibrés en droites gros sur les variétés symplectiques holomorphes projectives, répondant ainsi à une question posée par F. Charles.

1. Introduction

Zariski decomposition is a fundamental tool in the theory of linear series on algebraic surfaces. One possible generalization in higher dimension is the divisorial Zariski decomposition (see [44] in the projective case and [14] in the Kähler setting). When X is an irreducible symplectic manifold, by replacing the intersection form with the Beauville–Bogomolov–Fujiki quadratic form q_X on $H^2(X,\mathbb{C})$, Boucksom [14, Theorem 4.3 part (i), Proposition 4.4, Theorem 4.8 and Corollary 4.11] gave a characterization of his divisorial Zariski decomposition in terms of the quadratic form q_X (see Definition 3.4). We will refer to it as a *Boucksom–Zariski decomposition*. Several notions of singular irreducible symplectic varieties have received much attention in recent years, for several reasons. Let us simply mention here that the minimal model program naturally leads to singular minimal models. "Singular" irreducible symplectic varieties are studied in the theory of orbifolds or V-manifolds. They are also studied as moduli spaces of sheaves on $K3$ or abelian surfaces (see [47] and references to prior works therein). The period map and the moduli theory of "singular" irreducible symplectic varieties are extensively studied in [3, 4]. Finally, "singular" irreducible symplectic varieties appear as building blocks of mildly singular projective varieties with trivial canonical class (see [29, 20, 23]). See the recent survey [46] for all the different definitions and results. As the Boucksom–Zariski decomposition proved to be a very useful tool in the theory of smooth irreducible symplectic varieties it is natural to ask whether it holds for some classes of singular symplectic varieties. In this article, we show that the (Boucksom characterization of the divisorial) Zariski decomposition actually holds in the largest possible framework, namely it holds on any variety X with symplectic singularities (see Definition 2.1). In the smooth case, Boucksom actually proves it for pseudoeffective $\mathbb{R}$-classes, while here we restrict ourselves to effective $\mathbb{Q}$-divisors. To state our result recall that, following Kirchner [33], one can define a quadratic form $q_{X,\sigma}$ on $H^2(X,\mathbb{C})$, where σ is a reflexive 2-form on X, which is symplectic on X_{reg}. When X is a primitive symplectic variety (in the sense of Definition 2.2) and σ is a normalized symplectic form (see Definition 2.5), one shows that $q_{X,\sigma}$ is independent of σ and is called the Beauville–Bogomolov–Fujiki quadratic form (and denoted q_X). To deal with the non-$\mathbb{Q}$-factorial case we extend the definition of $q_{X,\sigma}$ to Weil divisors (cf. Definition 2.10). We refer the reader to Section 2.4 for further details.

Theorem 1.1. — *If X is a compact Kähler variety with symplectic singularities and $\sigma \in H^0(X,\Omega_X^{[2]})$ a reflexive 2-form, which is symplectic on X_{reg}, then all effective Weil $\mathbb{Q}$-divisors on X have a unique $q_{X,\sigma}$-Zariski decomposition, i.e., D can be written in a unique way as*

$$D = P(D) + N(D),$$

where $P(D)$ and $N(D)$ are effective Weil $\mathbb{Q}$-divisors satisfying the following.

1) *$P(D)$ is $q_{X,\sigma}$-nef, i.e., $q_{X,\sigma}(P(D),E) \geq 0$ for all effective Cartier divisors.*
2) *$N(D)$ is $q_{X,\sigma}$-exceptional, i.e., the Gram matrix $(q_X(N_i,N_j))_{i,j}$ of the irreducible components of the support of $N(D)$ is negative definite.*
3) *$q_{X,\sigma}(P(D),N(D)) = 0$.*
4) *For all integers $k \geq 0$ such that $kP(D)$ and kD are integral, then the natural map*

$$H^0(X,\mathcal{O}_X(kP(D))) \twoheadrightarrow H^0(X,\mathcal{O}_X(kD))$$

is a surjection.

In particular, if X is a primitive symplectic variety, any $\mathbb{Q}$-Cartier effective divisor has a unique q_X-Zariski decomposition with respect to the Beauville–Bogomolov–Fujiki quadratic form q_X.

The sheaves $\mathcal{O}_X(kP(D))$ and $\mathcal{O}_X(kD)$ above are the rank 1 reflexive sheaves associated to the Weil divisors $kP(D)$ and kD (see Section 2.3 for further details on Weil divisors and reflexive sheaves).

Our proof goes along the lines of Bauer's approach [5] to the Zariski decomposition for surfaces. As observed in [6], in order to have a Zariski-type decomposition on a compact Kähler variety it is sufficient to have a quadratic form on the second cohomology group (or on the divisor class group) that behaves like an intersection product, i.e., the intersection of two distinct prime divisors is always non-negative. It is hence enough to prove that on any compact Kähler variety X with symplectic singularities the quadratic form $q_{X,\sigma}$ introduced above behaves like an intersection product; see Theorem 3.10. Even if one wants to deal only with $\mathbb{Q}$-Cartier divisors, without further hypotheses these may a priori have non-$\mathbb{Q}$-Cartier irreducible components. Our approach to the Boucksom–Zariski decomposition requires to evaluate the quadratic form on those components. This is a technical difficulty that we can circumvent, but it renders the proof more involuted.

In the case of algebraic surfaces, as well as in the case of irreducible symplectic manifolds, the geometric significance of Zariski decompositions lies in the fact that, given a pseudoeffective integral divisor D with Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m > 1$ the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)),$$

i.e., all sections in $H^0(X, \mathcal{O}_X(mD))$ come from global sections of the "positive" part $\mathcal{O}_X(mP)$. The term "sufficiently divisible" here means that one needs to pass to a multiple mD that clears denominators in P for the statement to hold. In general, we do not know how to find numbers m for a given surface X and any integral pseudoeffective divisor. In [7], the main result tells us that the question about the possible values of m is strictly related to the bounded negativity conjecture, which is another open problem in the theory of linear series conjecturing the boundedness of negative self-intersections of irreducible divisors on any surface. Let us illustrate this phenomenon using the well-known case of smooth projective $K3$ surfaces. By the adjunction formula, for any irreducible and reduced curve C one has that $C^2 \geq -2$, and in order to clear denominators in the Zariski decomposition for any pseudoeffective divisors D, one can take $m = 2^{\rho-1}!$, where ρ denotes the Picard number – see [7, Example 3.2] for details.

It is natural to ask whether we can generalize the above considerations to the case of higher dimensional projective varieties. First of all, we have several variations on the classical Zariski decomposition, for instance the Cutkosky–Kawamata–Moriwaki–Zariski decomposition, but this decomposition, as was shown by Cutkosky [17], cannot exist in general. On the other hand, there is no natural and meaningful generalization of the bounded negativity conjecture in general. In [39], the authors constructed an example of a sequence of irreducible and reduced effective divisors D_k on smooth projective 3-fold Y such that $D_k^3 \to -\infty$.

However, as recalled before, we do have the Boucksom–Zariski decomposition on irreducible symplectic manifolds.

In the case of *projective* irreducible symplectic manifolds, similarly to the case of $K3$ surfaces, thanks to the presence and properties of the Beauville–Bogomolov quadratic form, we are able to prove an effective version of the analogue of the bounded negativity conjecture. More precisely in Proposition 4.8 we prove that for a projective irreducible symplectic manifold X the Beauville–Bogomolov self-intersection of any irreducible divisor is bounded from below by $4\,\mathrm{Card}(A_X)$, where $\mathrm{Card}(A_X)$ is the cardinality of the (finite) discriminant group

$$A_X := H^2(X,\mathbb{Z})^\vee / H^2(X,\mathbb{Z})$$

of the intersection lattice. The proof relies on two known results of Druel and Markman, which we recall in Propositions 4.6, 4.7. Notice that in the meantime these results have been proved in the singular framework, cf. [37, Theorem 1.2, item (1)] and [38, Remark 3.11], giving a lower bound on the Beauville–Bogomolov self-intersections of irreducible divisors in the singular case. For the boundedness of denominators in the Boucksom–Zariski decomposition we follow the lines of [7] to conclude with an effective bound on denominators.

Theorem 1.2. — *Let X be a smooth projective irreducible symplectic variety of Picard number $\rho(X)$. The denominators of the coefficients of the negative and positive parts of the Boucksom–Zariski decompositions of all pseudoeffective Cartier divisors are bounded by $(4\,\mathrm{Card}(A_X))^{\rho(X)-1}!$.*

For the proof see Corollary 4.10. Let us observe that $\rho(X) \leq h^{1,1}(X)$, and hence $(4\,\mathrm{Card}(A_X))^{h^{1,1}(X)-1}!$ gives a bound that is uniform for the whole family of deformations of X.

From Theorem 1.2 we easily deduce a result of effective birationality, which is interesting on its own.

Corollary 1.3. — *Let X be a smooth projective irreducible symplectic variety of dimension $2n$ and $L \in \mathrm{Pic}(X)$ a big line bundle on it. Then for all*

$$m \geq \frac{1}{2}(2n+2)(2n+3)(4\,\mathrm{Card}(A_X))^{\rho(X)-1}! \tag{1}$$

the map associated to the linear system $|mL|$ is birational onto its image.

Again, by replacing $\rho(X)$ with $h^{1,1}(X)$ in equation (1), we obtain an effective bound that holds for the whole family of deformations of X. In particular, the corollary above answers affirmatively and effectively to a strong version of a question asked by Charles in the Introduction in [16]. It is important to notice that even without answering this question Charles is able to obtain