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DISTINCTNESS OF TWO PSEUDO-ANOSOV MAPS

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ABSTRACT. — In 1981, Arnoux and Yoccoz gave the first examples of pseudo-Anosov maps with odd degree stretch factors. In 1985, D. Fried deduced the existence of a pseudo-Anosov map in genus three with the same stretch factor as the Arnoux–Yoccoz example in that genus, and asked if these were the same. We show that they are distinct. We do this by, in a sense, reversing Fried's construction; we show that the mapping torus of the pseudo-Anosov map induced by the Arnoux–Yoccoz map on the surface obtained by blowing up its two singularities has no cross section that is a torus with two points blown-up.

RÉSUMÉ (*Distinction de deux difféomorphisms pseudo-Anosov*). — En 1981 Arnoux et Yoccoz ont donné de premiers exemples de difféomorphisms pseudo-Anosov dont les coefficients de dilatation sont de degré impair. En 1985 D. Fried a déduit l'existence d'un difféomorphism pseudo-Anosov en genre trois ayant le même coefficient de dilatation que l'exemple en ce genre d'Arnoux-Yoccoz et a demandé si ces exemples sont les mêmes. Nous demontrons qu'ils ne le sont pas. On le fait, en gros, en reversant la construction de Fried ; nous montrons que le tore d'application du difféomorphism pseudo-Anosov, induit par celui d'Arnoux-Yoccoz sur l'éclatement de la surface aux deux singularités, n'admet pas de section qui soit l'éclatement en deux points d'un tore.

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1. Introduction

Thurston classified surface homeomorphisms into three classes, with the pseudo-Anosov homeomorphisms the main case of interest. As we recall below, associated to a pseudo-Anosov homeomorphism is an algebraic integer, its stretch factor. In the classic reference [7] for Thurston's work on surface homeomorphisms, published in 1979, the authors indicated that they had not yet seen a pseudo-Anosov map whose stretch factor is of algebraic degree greater than 2. In 1981, Arnoux and Yoccoz [2] gave examples of pseudo-Anosov maps with stretch factors of degree g for each $g \geq 2$. In a recent paper, Liechti–Strenner [21] write that this family of examples is "probably the single most widely studied" of the pseudo-Anosov maps; see [1, 4, 13, 16, 24, 25, 29] for some of this work.

D. Fried, in a series of influential papers including [8, 9, 10], used techniques of algebraic topology to study the dynamics of flows and especially of flows in 3-manifolds. As an illustration of some of this, Fried [11] in 1985 deduced the existence of a pseudo-Anosov map in genus three with the same stretch factor as the Arnoux–Yoccoz example in that genus, and asked if these were the same. We show that they are, in fact, distinct.

Whereas Arnoux–Yoccoz used methods of interval exchange transformations, Fried's approach gives the surface homeomorphism by blowing down the boundary circles of a particular cross section of the suspension flow on the mapping torus of the map, on the square torus blown up at two points, induced by a specific toral automorphism. If the two pseudo-Anosov maps were the same, then one could reverse Fried's procedure. That is, the square torus with two points blown up would appear as a cross section to the suspension flow on the mapping torus of the map induced by the Yoccoz–Arnoux map on the blow-up of the genus three surface at the two singular points of the Yoccoz–Arnoux map. We show that no cross section of this latter suspension flow has the Euler characteristic of a torus blow-up at two points, and thus deduce that the pseudo-Anosov maps of Arnoux–Yoccoz and of Fried cannot be the same.

THEOREM 1.1. — *The two pseudo-Anosov maps in genus three with stretch factor equal to the largest zero of $x^3 - x^2 - x - 1$ determined by Fried and by Arnoux–Yoccoz are distinct. In particular, there is no cross section to the suspension flow on the mapping torus of the blow-up of the Arnoux–Yoccox surface at its two singularities whose Euler characteristic equals to -2.*

We follow Fried's recipe for determining rough aspects of cross sections to the suspension flow in the mapping torus of a pseudo-Anosov map, in that we: (1) blow-up the singularities on the Arnoux–Yoccoz surface; (2) compute Fried's multivariable Lefschetz function for the suspension flow on the mapping torus of the induced pseudo-Anosov map; and (3) determine the specialization polynomials, each of which has degree giving the absolute value of the Euler characteristic of a corresponding section (and whose leading zero is the stretch factor of the pseudo-Anosov map given by the return map to the cross section). In Lemma 5.6, we use that information to show that there is no cross section whose Euler characteristic equals -2, and thus the theorem holds.

2. Background

2.1. Pseudo-Anosov maps as affine diffeomorphisms on translation surfaces. — The main case of the Dehn–Bers–Thurston classification of surface homeomorphisms is that of *pseudo-Anosov* maps; see [14] and [19] for much of the following. Suppose that S is an orientable closed real surface of genus $g \geq 2$. A homeomorphism $f : S \to S$ is called *pseudo-Anosov* if there exists a pair of invariant transverse measured (singular) foliations $(\mathcal{F}^u, \mu^u), (\mathcal{F}^s, \mu^s)$ and a real number $\lambda > 1$ such that f multiplies the transverse measure μ^u (resp. μ^s) by λ (resp. λ^{-1}). The real number λ, which Thurston [31] showed is always an algebraic integer, is called the *stretch factor* of the *pseudo-Anosov homeomorphism f*. One can extend this definition to the case with boundary; see Boyland's [6] discussion of 'standard models', where now the restriction is that the Euler characteristic of S be negative.

A pseudo-Anosov homeomorphism f is called *orientable* if $\mathcal{F}^u$ or $\mathcal{F}^s$ is orientable – that is, leaves can be consistently oriented. (It is well known that if either of these foliations is orientable, then so is the other.) As [20] recall (see their Theorem 2.4), a pseudo-Anosov homeomorphism f is orientable if and only if its stretch factor is an eigenvalue of the standard induced action on first homology $f_* : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z})$.

Hubbard–Masur [15] showed that the pair of measured foliations defines a quadratic differential and a complex structure on S, so that this quadratic differential is holomorphic. Orientability of the foliations corresponds to the quadratic differential being the square of a holomorphic 1-form (thus, an *abelian differential*), say ω. Fixing base points and integrating ω along paths defines local coordinates on S (in $\mathbb{C}$ or $\mathbb{R}^2$, depending on our need), with transition functions being translations, and the result is a *translation surface*, (S, ω). (The aforementioned singularities occur at the zeros of ω, and each has a cone angle that is a positive integral multiple of 2π.) The pseudo-Anosov f acts affinely with respect to the local Euclidean structure of (f, ω). It is thus an *affine diffeomorphism*; away from the singularities f is a diffeomorphism such that its Jacobian matrix with respect to the coordinates of the flat structure is constant. In fact, this constant matrix has eigenvalues λ, λ^{-1}. On the other hand, any affine diffeomorphism of a translation surface whose Jacobian matrix is of this form is a pseudo-Anosov homeomorphism.