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UNIFORM PARAMETERIZATION OF SUBANALYTIC SETS AND DIOPHANTINE APPLICATIONS

BY RAF CLUCKERS, JONATHAN PILA AND ALEX WILKIE

ABSTRACT. – We prove new parameterization theorems for sets definable in the structure $\mathbb{R}_{\text{an}}$ (i.e., for globally subanalytic sets) which are uniform for definable families of such sets. We treat both C^r-parameterization and (mild) analytic parameterization. In the former case we establish a polynomial (in r) bound (depending only on the given family) for the number of parameterizing functions. However, since uniformity is impossible in the latter case (as was shown by Yomdin via a very simple family of algebraic sets), we introduce a new notion, *analytic quasi-parameterization* (where many-valued complex analytic functions are used), which allows us to recover a uniform result.

We then give some diophantine applications motivated by the question as to whether the $H^{o(1)}$ bound in the Pila-Wilkie counting theorem can be improved, at least for certain reducts of $\mathbb{R}_{\text{an}}$. Both parameterization results are shown to give uniform $(\log H)^{O(1)}$ bounds for the number of rational points of height at most H on $\mathbb{R}_{\text{an}}$-definable Pfaffian surfaces. The quasi-parameterization technique produces the sharper result, but the uniform C^r-parameterization theorem has the advantage of also applying to $\mathbb{R}_{\text{an}}^{\text{pow}}$-definable families.

RÉSUMÉ. – Nous démontrons de nouveaux résultats de paramétrisations d'ensembles définissables dans $\mathbb{R}_{\text{an}}$ (aussi appelés ensembles sous-analytiques globaux), uniformément dans les familles définissables. Nous traitons les paramétrisations C^r ainsi que les paramétrisations douces et analytiques. Dans le cas C^r, nous obtenons une borne polynômiale (en r, et dépendant seulement de la famille) pour le nombre de fonctions paramétrisantes. Dans le cas de paramétrisations analytiques, comme l'uniformité est impossible (démontré par Yomdin pour une famille semi-algébrique très simple), nous introduisons une nouvelle notion de *paramétrisations quasi-analytiques* (utilisant les fonctions analytiques complex multi-valuées), ce qui nous permet d'obtenir des résultats uniformes. Ensuite nous donnons des applications diophantiennes motivées par la question de savoir si la borne $H^{o(1)}$ dans le théorème de comptage de Pila-Wilkie peut être améliorée pour certaines réductions de la structure $\mathbb{R}_{\text{an}}$. Nos deux approches de paramétrisations nous permettent d'obtenir des bornes uniformes de grandeur $(\log H)^{O(1)}$ pour le nombre de points rationels de hauteur au maximum H sur les surfaces pfaffiennes

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qui sont $\mathbb{R}_{\mathrm{an}}$-définissables. Les paramétrisations quasi-analytiques nous donnent des résultats plus fins, mais les paramétrisations C^r ont l'avantage de fonctionner aussi dans le cadre plus général de familles $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$-définissables.

1. Introduction

The aim of this section is to give an informal account of the results appearing in this paper. Precise definitions and statements are given in the next section.

So, we are concerned with *parameterizations* of bounded definable subsets of real euclidean space. The definability here is with respect to some fixed (and, for the moment, arbitrary) o-minimal expansion of the real field. By a parameterization of such a set $X \subseteq \mathbb{R}^n$, we mean a finite collection of definable maps from $(0,1)^m$ to $\mathbb{R}^n$, where $m := \dim(X)$, whose ranges cover X. The fact that parameterizations always exist is an easy consequence of the cell decomposition theorem, but the aim is to construct them with certain differentiability conditions imposed on the parameterizing functions together with bounds on their derivatives. The first result in this generality was obtained in [27] (by adapting methods of Yomdin [33] and Gromov [13] who dealt with the semi-algebraic case), where it was shown that for each positive integer r there exists a parameterization consisting of C^r functions all of whose derivatives (up to order r) are bounded by 1. Further, the parameterizing functions may be found *uniformly*. This means that if $\mathcal{X} = \{X_t : t \in T\}$ is a definable family of m-dimensional subsets of $(0,1)^n$ (say), i.e., the relation "$t \in T$ and $x \in X_t$" is definable in both x and t, then there exists a positive integer N_r such that for each $t \in T$, at most N_r functions are required to parameterize X_t and each such function is definable in t. (The bound N_r does, of course, also depend on the family $\mathcal{X}$, but we usually suppress this in the notation. The point is that it is independent of t.) Unfortunately, the methods of [27] do not give an explicit bound for N_r and it is the first aim of this paper to do so in the case that the ambient o-minimal structure is the restricted analytic field $\mathbb{R}_{\mathrm{an}}$ (where the bounded definable sets are precisely the bounded subanalytic sets), or a suitable reduct of it. We prove, in this case, that N_r may be taken to be a polynomial in r (which depends only on the given family $\mathcal{X}$). While we have only diophantine applications in mind here, this result already gives a complete answer to an open question, raised by Yomdin, coming from the study of entropy and dynamical systems (see e.g., [33], [32], [13], [3]). In fact, even in the case that the ambient structure is just the ordered field of real numbers (which is certainly a suitable reduct of $\mathbb{R}_{\mathrm{an}}$ to which our result applies), the polynomial bound appears to be new and, indeed, gives a partial answer to a question raised in [3] (just below Remark 3.8); the essential missing ingredient to solve this question completely is an effective form of the preparation result of [20] in the semi-algebraic case. Our uniform C^r-parameterization theorem also holds for the expansion of $\mathbb{R}_{\mathrm{an}}$ by all power functions (i.e., the structure usually denoted $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$) and suitable reducts (to be clarified in Section 2) of it. In fact, we obtain a pre-parameterization result in Section 4.2 which underlies C^r-parameterizations.

Next we consider *mild* parameterizations. Here it is more convenient to consider parameterizing functions with domain $(-1,1)^m$ (where m is the dimension of the set being parameterized) and we demand that they are C^∞ and we put a bound on *all* the derivatives. We shall only be concerned with functions that satisfy a so called 0-mild condition, namely that there exists an $R > 1$ such that for each positive integer d, all their d'th derivatives have a bound of order $R^{-d} \cdot d!$ (which in fact forces the functions to be real analytic). It was shown in [15] that any reduct of $\mathbb{R}_{\mathrm{an}}$ has the 0-mild parameterization property: every definable subset of $(-1,1)^n$ has a parameterization by a finite set of 0-mild functions. However, this result cannot be made uniform. For Yomdin showed in [34, Proposition 3.3] (see also [35, page 416]) that the number of 0-mild functions required to parameterize the set $\{\langle x_1, x_2\rangle \in (-1,1)^2 : x_1 \cdot x_2 = t\}$ necessarily tends to infinity as $t \to 0$. Our second parameterization result recovers uniformity in the 0-mild setting but at the expense of, firstly, covering larger sets than, but ones having the same dimension as, the sets in the given family and secondly, covering not by ranges of 0-mild maps but by solutions to (a definable family of) Weierstrass polynomials with 0-mild functions as coefficients.

In [27] the parameterization theorem is applied to show that any definable subset of $(0,1)^n$ (the ambient o-minimal structure being, once again, arbitrary) either contains an infinite semi-algebraic subset or else, for all $H \geq 1$, contains at most $H^{o(1)}$ rational points whose coordinates have denominators bounded by H. (For the purposes of this introduction we refer to such points as H*-bounded* rational points.) Although this result is best possible in general, and is so even for one dimensional subsets of $(0,1)^2$ definable in the structure $\mathbb{R}_{\mathrm{an}}$, it has been conjectured that the $H^{o(1)}$ bound may be improved to $(\log H)^{O(1)}$ for certain reducts of $\mathbb{R}_{\mathrm{an}}$ (specifically, for sets definable from restricted *Pfaffian* functions), and it is our final aim in this paper to take a small step towards such a conjecture.

We first observe that the point counting theorem from [27] quoted above follows (by induction on dimension) from the following uniform result (the main lemma of [27] on page 610). Namely, if $m < n$ and $\mathcal{X} = \{X_t : t \in T\}$ is a definable family of m-dimensional subsets of $(0,1)^n$, and $\varepsilon > 0$, then there exists a positive integer $d = d(\varepsilon, n)$ such that for each $t \in T$ and for all $H \geq 1$, all the H-bounded rational points of X_t are contained in the union of at most $O(H^\varepsilon)$ algebraic hypersurfaces of degree at most d, where the implied constant depends only on $\mathcal{X}$ and ε. Now, for the structure $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$ (or any of its suitable reducts), our uniform C^r-parameterization theorem allows us to improve the bound here on the number of hypersurfaces to $O((\log H)^{O(1)})$ (for $H > e$, with the implied constants depending only on the family $\mathcal{X}$) but, unfortunately, their degrees have this order of magnitude too. Actually, the bound on the degrees is completely explicit, namely $[(\log H)^{m/(n-m)}]$, but as this tends to infinity with H, the inductive argument used in [27] (where the degree d only depended on ε and n) breaks down at this point. Our 0-mild (quasi-) parameterization theorem does give a better result for (suitable reducts of) the structure $\mathbb{R}_{\mathrm{an}}$ in that the number of hypersurfaces is bounded by a constant (depending only on $\mathcal{X}$), but the bound for their degrees is the same as above and so, once again, the induction breaks down.

We can, however, tease out a uniform result for rational points on certain one and two dimensional sets definable from restricted Pfaffian functions, but for the general conjecture a completely new uniform parameterization theorem that applies to the intersection of a definable set of constant complexity with an algebraic hypersurface of *nonconstant* degree is badly needed.