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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

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REGULARITY OF WEAK MINIMIZERS OF THE K-ENERGY AND APPLICATIONS TO PROPERNESS AND K-STABILITY

BY ROBERT J. BERMAN, TAMÁS DARVAS AND CHINH H. LU

ABSTRACT. – Let (X, ω) be a compact Kähler manifold and $\mathcal{H}$ the space of Kähler metrics cohomologous to ω. If a csck metric exists in $\mathcal{H}$, we show that all finite energy minimizers of the extended K-energy are smooth csck metrics, partially confirming a conjecture of Y.A. Rubinstein and the second author. As an immediate application, we obtain that the existence of a csck metric in $\mathcal{H}$ implies J-properness of the K-energy, thus confirming one direction of a conjecture of Tian. Exploiting this properness result we prove that an ample line bundle (X, L) admitting a csck metric in $c_1(L)$ is K-polystable. When the automorphism group is finite, the properness result, combined with a result of Boucksom-Hisamoto-Jonsson, also implies that (X, L) is uniformly K-stable.

RÉSUMÉ. – Soient (X, ω) une variété kählérienne compacte et $\mathcal{H}$ l'espace des métriques de Kähler dans la classe de cohomologie de ω. S'il existe une métrique csck dans $\mathcal{H}$, nous montrons que tous les minimiseurs dans l'espace d'énergie finie de la fonctionnelle K-énergie sont lisses, confirmant partiellement une conjecture de Y.A. Rubinstein et le deuxième auteur. Comme une conséquence immédiate, nous en déduisons que l'existence d'une métrique csck dans $\mathcal{H}$ implique la J-propreté de la fonctionnelle K-énergie. Ceci confirme une direction de la conjecture de Tian. En utilisant cette propreté nous montrons qu'un fibré ample (X, L) admettant une métrique csck dans $c_1(L)$ est K-polystable. Quand le groupe d'automorphisme est fini le résultat de propreté, combiné avec un résultat récent de Boucksom-Hisamoto-Jonsson, implique aussi que (X, L) est uniformément K-stable.

1. Introduction and main results

Let (X, J, ω) be a compact connected Kähler manifold. By

$$\mathcal{H}_\omega = \{v \in \mathcal{C}^\infty(X) \mid \omega_v := \omega + i\partial\bar{\partial}v > 0\}$$

we denote the space of Kähler potentials. By the $\partial\bar{\partial}$-lemma of Hodge theory, up to a constant, this space is in a one-to-one correspondence with $\mathcal{H}$, the space of Kähler metrics cohomologous to ω. The problem of finding canonical metrics in $\mathcal{H}$ goes back to Calabi in the fifties. In this work we will point necessary conditions under which $\mathcal{H}$ admits constant scalar curvature Kähler (csck) metrics, in terms of energy properness.

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We now elaborate on the terminology necessary to state our main results. To have a one-to-one correspondence between potentials and metrics, we consider the space

$$\mathcal{H}_0 := \mathcal{H}_\omega \cap \mathrm{AM}^{-1}(0),$$

and we always work on the level of potentials unless specified otherwise (for the definition of AM see (3) below). The connected Lie group of holomorphic automorphisms

$$G := \mathrm{Aut}_0(X, J)$$

acts naturally on $\mathcal{H}$ via pullbacks, hence it also acts on $\mathcal{H}_0$ (see [22, Section 5.2] for a precise description of this action on the level of potentials).

Motivated by results and ideas in conformal geometry, in the 90's Tian introduced the notion of "J-properness" on $\mathcal{H}_\omega$ [45, Definition 5.1] in terms of Aubin's nonlinear energy functional J_ω and the Mabuchi K-energy E. This condition says that for any $u_j \in \mathcal{H}_\omega$ we have

$$J_\omega(u_j) \to \infty \quad \text{implies} \quad E(u_j) \to \infty. \tag{1}$$

We refer to Section 2 for the precise definitions of J_ω and E.

Tian conjectured that existence of constant scalar curvature Kähler (csck) metrics in $\mathcal{H}_\omega$ should be equivalent to J-properness of the K-energy E [45, Remark 5.2],[47] and this was proved for Fano manifolds with G trivial [46, 50]. In [36, Theorem 1] the "strong form" of the J-properness condition (1) was obtained, confirming another conjecture of Tian from [46] (for Fano manifold with trivial G), saying that the K-energy grows at least linearly with respect to the J-functional. This stronger form has been later adopted in the literature, sometimes referred to as "coercivity".

When G is non-trivial it was known that the conjecture cannot, in general, hold as stated above and numerous modifications were proposed by Tian (see [47, Conjecture 7.12], [48]). In [22], Y.A. Rubinstein and the second named author disproved one of these conjectures, proved the remaining ones for general Fano manifolds, and the following conjecture was stated for general Kähler manifolds:

CONJECTURE 1.1 (Conjecture 2.8 in [22]). – *Suppose (X, ω) is a Kähler manifold. There exists a csck metric cohomologous to ω if and only if for some $C, D > 0$ we have*

$$E(u) \geq C \inf_{g \in G} J_\omega(g.u) - D, \ \ u \in \mathcal{H}_\omega.$$

This "modified properness conjecture" thus reduces to Tian's original prediction in case G is trivial and was originally stated for Fano manifolds by Tian himself [48]. It was proved in this context (of Fano manifolds) in [22, Theorem 2.4], and this paper also linked the resolution of the general conjecture to a regularity question on weak minimizers of the K-energy that we elaborate now.

We denote by $(\mathcal{E}^1, d_1)$ the metric completion of $\mathcal{H}_\omega$ with respect to the L^1-type Mabuchi path length metric d_1. We refer to Sections 2.1-2.2 for more precise details about this metric structure introduced in [19]. The point of connection with the questions investigated here is the fact that d_1 metric growth is comparable to J_ω [22, Proposition 5.5], and we refer to [22, Section 4, Section 5] for a more detailed exposition on how the d_1-metric geometry relates to J-properness. Let us now state the regularity conjecture of [22] (see [22, Conjecture 2.9]) and the theorem that connects it to Conjecture 1.1 above:

CONJECTURE 1.2 (Conjecture 2.9 in [22]). – *Suppose* (X, ω) *is a compact Kähler manifold. The minimizers of the extended K-energy* $E : \mathcal{E}^1 \to (-\infty, +\infty]$ *are smooth csck metrics.*

THEOREM 1.3 (Theorem 2.10 in [22]). – *Conjecture 1.2 implies Conjecture 1.1.*

Our first main result partially confirms Conjecture 1.2 and also a less general conjecture of X.X. Chen [17, Conjecture 6.3]:

THEOREM 1.4. – *Suppose* (X, ω) *is a csck manifold. If* $v \in \mathcal{E}^1$ *minimizes the extended K-energy* $E : \mathcal{E}^1 \to (-\infty, +\infty]$*, then* v *is a smooth csck potential. In particular there exists* $g \in G$ *such that* $g^*\omega_v = \omega$.

The last claim follows from the uniqueness result of [8]. Using this result and Theorem 1.3 we immediately obtain one direction of Conjecture 1.1:

THEOREM 1.5. – *Suppose* (X, ω) *is a csck manifold. Then for some* $C, D > 0$ *we have*

$$E(u) \geq C \inf_{g \in G} J_\omega(g.u) - D, \quad u \in \mathcal{H}_\omega. \tag{2}$$

The proof of Theorem 1.4 relies on the L^1-Mabuchi geometry of $\mathcal{H}_\omega$ introduced in [20, 19], the finite energy pluripotential theory of [9, 28] and the convexity methods of [22, 12] and [8]. Realizing that the metric geometry of $\mathcal{H}_\omega$ and J-properness should be related seems to have first appeared in [17, Conjecture 6.1], but this work rather proposed the use of the L^2-Mabuchi metric on $\mathcal{H}_\omega$.

As a consequence of Theorem 1.5 and the techniques of [46, 6] we obtain a result on K-polystability, originally proved by Mabuchi ([33, Main Theorem] see also [32]), using a completely different argument. Slightly less general, or different flavor results were obtained by Stoppa, Stoppa-Székelyhidi, Székelyhidi [40, Theorem 1.2], [41, Theorem 1.4], [43, Theorem A] and others. We recall the relevant terminology in the last section of the paper.

THEOREM 1.6. – *Suppose* $L \to X$ *is a positive line bundle. If there exists a csck metric in the class* $c_1(L)$*, then* (X, L) *is K-polystable.*

The idea of proving K-stability via properness goes back to Tian's seminal paper [46]. The main point of our approach, involving geodesic rays, is to generalize the findings of [6] from the Fano case.

In case the group G is trivial, the results in [11, 13, 24] show that properness implies uniform K-stability in the L^1-sense (for terminology, see [11, 13, 24] and references therein). Thus, as a consequence of Theorem 1.5 we obtain the following:

COROLLARY 1.7. – *Assume that* (X, L) *is a positive line bundle and* G *is trivial. If there exists a csck metric in* $c_1(L)$*, then* (X, L) *is uniformly K-stable.*