ISSN 0012-9593

quatrième série - tome 53 fascicule 3 mai-juin 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 53, 2020, p. 559 à 588

FINITENESS OF PARTIALLY HYPERBOLIC ATTRACTORS WITH ONE-DIMENSIONAL CENTER

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Abstract. – We prove that the set of diffeomorphisms having at most finitely many attractors contains a dense and open subset of the space of C^1 partially hyperbolic diffeomorphisms with one-dimensional center.

This is obtained thanks to a robust geometric property of the stable and unstable laminations that we show to hold after perturbations of the dynamics. This technique also allows to prove that C^1-generic diffeomorphisms far from homoclinic tangencies in dimension 3 either have at most finitely many attractors, or satisfy Newhouse phenomenon.

Résumé. – Nous montrons que l'ensemble des difféomorphismes ayant un nombre au plus fini d'attracteurs contient un ouvert dense de l'espace des difféomorphismes C^1 partiellement hyperboliques avec fibré central de dimension 1.

Ce résultat découle d'une propriété géométrique robuste des laminations stables et instables, qui peut être obtenue par perturbation de la dynamique. Cette technique nous permet également de montrer que sur les variétés de dimension 3, les difféomorphismes C^1-génériques loin des tangences homoclines ou bien ont un nombre au plus fini d'attracteurs, ou bien présentent le phénomène de Newhouse.

1. Introduction

A main question when one studies the qualitative properties of a dynamical system consists in describing its attractors. More generally, one studies how the dynamics decomposes into elementary invariant pieces. This is for instance the purpose of Smale's spectral decomposition theorem for hyperbolic dynamics. This paper discusses the number of attractors for diffeomorphisms f of a compact boundaryless manifold under a weaker hyperbolicity property.

The authors were partially supported by the Balzan Research Project of J. Palis. R.P. and M.S were partially supported by CSIC group 618, IFUM, CNRS and MathAmSud:Physeco. R.P. was also partially supported by the Laboratoire Mathematique d'Orsay. S. C. was partially supported by IFUM and the ERC project 692925 *NUHGD*..

 doi:10.24033/asens.2428

One usually defines an *attractor* of f as an f-invariant non-empty compact set K which admits a neighborhood U satisfying $K = \bigcap_{n\in\mathbb{N}} f^n(U)$ and which is transitive (i.e., the dynamics of f on K contains a dense forward orbit). An attractor which is reduced to a finite set is called a *sink*. In general a diffeomorphism may have no attractors (this is for instance the case of the identity) and one introduces a weaker notion: a *quasi-attractor* of f is an f-invariant non-empty compact set which has the following two properties:

— K admits a basis of open neighborhoods U such that $f(\overline{U}) \subset U$,

— K is chain-transitive, i.e., for any $\varepsilon > 0$ there exists a dense sequence $(x_n)_{n\geq 0}$ in K which satisfies $d(f(x_n), x_{n+1}) < \varepsilon$ for each $n \geq 0$.

Any homeomorphism of a compact metric space admits at least one quasi-attractor. For hyperbolic diffeomorphisms they coincide with usual attractors. For any diffeomorphisms in a dense G_δ-set of $\mathrm{Diff}^1(M)$, the set of points whose positive orbit accumulates on a quasi-attractor is a dense G_δ-subset of M, see [4].

The number of attractors may be infinite for large classes of dynamical systems. This is the case near the set $\mathcal{T}$ of diffeomorphisms exhibiting a homoclinic tangency, i.e., which have a hyperbolic periodic orbit whose stable and unstable manifolds are not transverse: this has been proved by Newhouse [22] inside the space $\mathrm{Diff}^2(M)$ of C^2 diffeomorphisms of a surface M, or in $\mathrm{Diff}^1(M)$ when $\dim(M) \geq 3$, under a stronger assumption on the homoclinic tangency, see for instance [6, 7, 3, 10]. In fact, all the known abundant classes of diffeomorphisms are in the limit of diffeomorphisms exhibiting a homoclinic tangency. This motivated the following conjecture [24, 3], see also [11].

Conjecture (Bonatti, Palis). – *There exists a dense and open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}}$ such that the diffeomorphism $f \in \mathcal{U}$ has at most finitely many quasi-attractors (and attractors).*

More generally, one may consider the chain-recurrence classes of diffeomorphisms [4, 10], which decompose the chain-recurrent dynamics. Bonatti has conjectured [3] that for diffeomorphisms in $\mathcal{U}$, the number of chain-recurrence classes is finite.

On surfaces, this conjecture is implied by a stronger result, proved by Pujals and Sambarino [29]. This paper is a step towards this conjecture when M has dimension 3 and in some regions of $\mathrm{Diff}^1(M)$, when M has dimension larger than 3. These results were announced in [11] and [12].

We consider the (open) subset $\mathrm{PH}^1_{c=1}(M)$ of C^1-diffeomorphisms f of M which preserve a *partially hyperbolic* decomposition, with a one-dimensional center, i.e., which preserve a splitting $TM = E^s \oplus E^c \oplus E^u$, $\dim(E^c) = 1$, with the property that for some $\ell > 0$ and for every unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) we have that:

$$\|Df^\ell_x v^s\| < \min\{1, \|Df^\ell_x v^c\|\} \leq \max\{1, \|Df^\ell_x v^c\|\} < \|Df^\ell_x v^u\|. \tag{1.1}$$

We will always assume that both E^s, E^u are non-trivial. Partial hyperbolicity has been playing a central role in the study of differentiable dynamics due to its robustness and how it is related with the absence of homoclinic tangencies (see [10, 15]). It also prevents the existence of sinks.

Under some global assumptions it is sometimes possible to show that partially hyperbolic dynamics with one-dimensional center have finiteness and sometimes even uniqueness of quasi-attractors (see e.g., [9, 25], [19, Section 6.2] or [25, Section 5]). However, it is easy to construct examples of partially hyperbolic diffeomorphisms with infinitely many quasi-attractors (e.g., by perturbing Anosov×Identity on $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$). Here, we prove that this is a fragile situation:

Theorem A. – *There exists an open and dense subset $\mathcal{O}$ of $\mathrm{PH}^1_{c=1}(M)$ such that every $f \in \mathcal{O}$ has at most finitely many quasi-attractors.*

In dimension 3, we obtain a stronger conclusion:

Theorem B. – *Let M be a 3-dimensional manifold. There is an open and dense subset $\mathcal{U} \subset \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}}$ of diffeomorphisms f such that:*

- *either f has at most finitely many quasi-attractors,*
- *or f is accumulated by diffeomorphisms with infinitely many sinks.*

Another work [14] will address the finiteness of the set of sinks for diffeomorphisms far from homoclinic tangencies and will conclude the proof of Bonatti-Palis Conjecture in dimension 3. We emphasize that this corresponds to a problem of different nature.

More generally we consider invariant compact sets Λ which are *partially hyperbolic*, i.e., which admit a continuous Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ and $\ell > 0$ with the property that for every unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) the property (1.1) holds. Theorem A is a consequence of a more precise result:

Theorem C. – *There exists a dense G_δ subset $\mathcal{G}_1$ of $\mathrm{Diff}^1(M)$ with the following property. Consider $f_0 \in \mathcal{G}_1$ and a compact set $U \subset M$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f_0^n(U)$ is a partially hyperbolic set with one-dimensional center.*

Then, for every f C^1-close to f_0 the set U contains at most finitely many quasi-attractors of f.

As a consequence we obtain a (weak) version of an unpublished theorem by Bonatti-Gan-Li-Yang ([8]).

Corollary D. – *There exists a dense G_δ subset $\mathcal{G}_2$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{G}_2$ and Q is a partially hyperbolic quasi-attractor for f with one dimensional center, then Q is not accumulated by other quasi-attractors.*

Such quasi-attractors are called *essential attractors* in [8] since it follows from their properties that their basin contains a residual subset in an attracting neighborhood. In [8] they prove that *every* quasi-attractor for a C^1-generic diffeomorphism C^1-far from homoclinic tangencies is an essential attractor.