ISSN 0012-9593

quatrième série - tome 53 fascicule 5 septembre-octobre 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 53, 2020, p. 1105 à 1119

AN INTRINSIC CHARACTERIZATION OF C*-SIMPLICITY

BY Matthew KENNEDY

ABSTRACT. – A group is said to be C*-simple if its reduced C*-algebra is simple. We establish an intrinsic (group-theoretic) characterization of groups with this property. Specifically, we prove that a discrete group is C*-simple if and only if it has no non-trivial amenable uniformly recurrent subgroups. We further prove that a group is C*-simple if and only if it satisfies an averaging property considered by Powers.

RÉSUMÉ. – Un groupe est dit C*-simple si son C*-algèbre réduite est simple. Nous établissons une caractérisation intrinsèque (groupe-théorique) des groupes ayant cette propriété. Spécifiquement, nous prouvons qu'un groupe est C*-simple si et seulement s'il n'a pas de sous-groupes non triviaux uniformément récurrents. Nous démontrons en outre qu'un groupe est C*-simple si et seulement s'il satisfait une propriété de moyennage considérée par Powers.

1. Introduction

A group is said to be C*-simple if its reduced C*-algebra is simple, meaning that it has no non-trivial proper closed two-sided ideals. It has been an open problem for some time to determine an intrinsic group-theoretic characterization of groups that are C*-simple, along the lines of Murray and von Neumann's characterization of groups that give rise to factorial von Neumann algebras as groups with the infinite conjugacy class property.

It is not difficult to see that a C*-simple group necessarily has no non-trivial normal amenable subgroups, and based on a great deal of accumulated evidence, it was thought that this condition might be sufficient. However, in a major breakthrough, Le Boudec [17] constructed examples showing that this is not the case.

The main result in this paper is an intrinsic group-theoretic characterization of C*-simplicity in terms of the nonexistence of non-trivial amenable subgroups satisfying a condition that is weaker than normality. We say that a subgroup H of a discrete group G is *residually normal* if there exists a finite subset $F \subset G \setminus \{e\}$ such that $F \cap gHg^{-1} \neq \emptyset$ for all $g \in G$.

Author supported by NSERC Grant Number 418585.

 doi:10.24033/asens.2441

Theorem 1.1. – *A discrete group is C*-simple if and only if it has no amenable residually normal subgroups.*

We prove this result by analyzing the dynamics of the conjugation action of a group on its space of subgroups and invoking the dynamical characterization of C*-simplicity from [15]. The study of this action underlies the theory of uniformly recurrent subgroups introduced by Glasner and Weiss [8]. Using their terminology, Theorem 1.1 is equivalent to the following result (see Section 4).

Theorem 1.2. – *A discrete group is C*-simple if and only if it has no non-trivial amenable uniformly recurrent subgroups.*

The notion of a uniformly recurrent subgroup can be viewed as a topological analogue of the measure-theoretic notion of an invariant random subgroup introduced by Abért, Glasner and Virág [1]. Many rigidity results in ergodic theory, and in particular results about the rigidity of characters on groups, can be viewed as results about the non-existence of certain invariant random subgroups. From this perspective, Theorem 1.2 can be viewed as a kind of rigidity phenomenon in topological dynamics.

The theory of C*-simplicity began with Powers' theorem [19] stating that free groups on two or more generators are C*-simple. The key insight underlying Powers' proof is that the left regular representation of these groups satisfies a very strong averaging property.

Definition 1.3. – A discrete group G is said to have *Powers' averaging property* if for every element a in the reduced C*-algebra $C^*_\lambda(G)$ and every $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i} a \lambda_{g_i}^{-1} - \tau_\lambda(a) 1 \right\| < \epsilon,$$

where τ_λ denotes the canonical tracial state on $C^*_\lambda(G)$.

It is straightforward to show that any group satisfying Powers' averaging property is C*-simple. In fact, prior to the publication of [15] and [2], essentially the only method for establishing the C*-simplicity of a given group was to show, often with great difficulty, that the group satisfied some variant of Powers' averaging property.

The next (perhaps somewhat surprising) result demonstrates the remarkable depth of Powers' insight. It turns out that every C*-simple group necessarily satisfies Powers' averaging property.

Theorem 1.4. – *A discrete group is C*-simple if and only if it satisfies Powers' averaging property.*

In addition to this introduction there are five other sections. In Section 2 we briefly review preliminary material. In Section 3 we clarify the relationship between C*-simplicity and the unique trace property, and obtain some technical results about the dual space of the reduced C*-algebra of a discrete group. In Section 4 we prove the characterization of C*-simplicity in terms of uniformly recurrent subgroups. In Section 5 we prove the main result characterizing C*-simplicity in terms of amenable residually normal subgroups. Finally, in Section 6 we prove that a group is C*-simple if and only if it has Powers' averaging property.

New developments

Since the first draft of this paper appeared in September 2015, a number of related developments have occurred. First, Haagerup [10] independently obtained Theorem 1.4.

Second, Theorem 1.2 has been applied by Le Boudec and Matte Bon to study the C*-simplicity of groups of homeomorphisms of the circle, and in particular Thompson's groups F, T and V. They proved that Thompson's group V is C*-simple, and proved that the non-amenability of Thompson's group F is equivalent to the C*-simplicity of T.

Third, Bryder and the author [5] applied similar ideas to study the maximal ideals of reduced (twisted) crossed products over C*-simple groups. In particular, we established a bijective correspondence between maximal ideals of the underlying C*-algebra and maximal ideals of the reduced crossed product.

Finally, Kawabe [16] extended the methods introduced in this paper to undertake a systematic study of the ideal structure of reduced crossed products. In particular, he obtains necessary and sufficient conditions for a commutative C*-algebra equipped with an action of a discrete group to separate ideals in the corresponding reduced crossed product. We also mention a recent paper of Bryder [4] that applies similar ideas to investigate the structure of reduced crossed products of noncommutative C*-algebras.

Acknowledgements

The author is grateful to Adrien Le Boudec, Emmanuel Breuillard, Pierre-Emmanuel Caprace, Kenneth Davidson, Eli Glasner, Mehrdad Kalantar, Narutaka Ozawa and Sven Raum for many helpful comments and suggestions.

2. Preliminaries

2.1. The reduced C*-algebra

Let G be a discrete group with identity element e. Let λ denote the left regular representation of G on the Hilbert space $\ell^2(G)$. The *reduced C*-algebra* $\mathrm{C}^*_\lambda(G)$ is the norm closed algebra generated by the image of G under λ.

Let $\{\delta_g \mid g \in G\}$ denote the standard orthonormal basis for $\ell^2(G)$. Then every element $a \in \mathrm{C}^*_\lambda(G)$ has a Fourier expansion

$$a = \sum_{g \in G} \alpha_g \lambda_g,$$

uniquely determined by $\alpha_g = \langle a\delta_e, \delta_g\rangle$ for $g \in G$.

A linear functional ϕ on $\mathrm{C}^*_\lambda(G)$ is said to be a *state* if it is unital and positive, i.e., $\phi(1) = 1$ and $\phi(a) \geq 0$ for every $a \in \mathrm{C}^*_\lambda(G)$ with $a \geq 0$. If, in addition, $\phi(ab) = \phi(ba)$ for all $a, b \in \mathrm{C}^*_\lambda(G)$, then ϕ is said to be *tracial*. The C*-algebra $\mathrm{C}^*_\lambda(G)$ is always equipped with a *canonical tracial state* τ_λ defined by $\tau_\lambda(a) = \langle a\delta_e, \delta_e\rangle$.

For general facts about group C*-algebras, crossed products and completely positive maps, we refer the reader to Brown and Ozawa's book [3].