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SIEVE WEIGHTS AND THEIR SMOOTHINGS

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Abstract. – We obtain asymptotic formulas for the $2k$th moments of partially smoothed divisor sums of the Möbius function. When $2k$ is small compared with A, the level of smoothing, then the main contribution to the moments comes from integers with only large prime factors, as one would hope for in sieve weights. However if $2k$ is any larger, compared with A, then the main contribution to the moments comes from integers with quite a few prime factors, which is not the intention when designing sieve weights. The threshold for "small" occurs when $A = \frac{1}{2k}\binom{2k}{k} - 1$.

One can ask analogous questions for polynomials over finite fields and for permutations, and in these cases the moments behave rather differently, with even less cancelation in the divisor sums. We give, we hope, a plausible explanation for this phenomenon, by studying the analogous sums for Dirichlet characters, and obtaining each type of behavior depending on whether or not the character is "exceptional".

Résumé. – On obtient des formules asymptotiques pour les $2k$-ièmes moments de quelques sommes partiellement lissées de la fonction de Möbius sur les diviseurs d'un entier. Quand $2k$ est petit en comparaison avec A, qui est le niveau de lissage, alors la contribution principale aux moments provient des entiers n'ayant que de grands facteurs premiers, comme on l'espérait pour un poids de crible. Cependant, si $2k$ est plus grand en comparaison avec A, alors la contribution principale aux moments provient des entiers ayant beaucoup de facteurs premiers, ce qui n'est pas l'intention quand on crée des poids de crible. La valeur seuil pour « petit » est $A = \frac{1}{2k}\binom{2k}{k} - 1$.

On peut aussi poser des questions analogues pour les polynômes sur des corps finis et pour les permutations, et dans ces cas les moments se comportent de façon assez différente, avec moins d'annulations dans les sommes de diviseurs. On donne, on espère, une explication plausible pour ce phénomène, en étudiant les sommes analogues pour les caractères de Dirichlet, et en obtenant chaque type de comportement selon le caractère « exceptionnel » ou non.

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1. Introduction

Sieve methods are a set of techniques which give upper and lower bounds for the number of elements of a set of integers $\mathscr{A}$ which have no 'small' prime factors. Their key benefit is that they are very flexible - one can obtain bounds of the correct order of magnitude for many interesting sets $\mathscr{A}$, even though obtaining asymptotic formulae looks completely hopeless. In particular, they are typically very effective at obtaining upper bounds for the number of primes in sets $\mathscr{A}$ of interest which are only worse than the conjectured truth by a constant factor.

One of the most important sieves is the Selberg sieve. Selberg's approach [19] starts with the inequality

$$\Big(\sum_{\substack{d|n\\ P^+(d)\le z}} \lambda_d\Big)^2 \ge \sum_{\substack{d|n\\ P^+(d)\le z}} \mu(d) = \begin{cases}1, & P^-(n)\ge z,\\ 0, & \text{otherwise,}\end{cases} \tag{1.1}$$

which is valid for any real numbers λ_d with $\lambda_1 = 1$. Here $P^+(n)$ and $P^-(n)$ are the largest and smallest prime factors of n respectively. Summing (1.1) over $n\in\mathscr{A}$ gives

$$\begin{aligned}\#\{n\in\mathscr{A} : P^-(n)\ge z\} &\le \sum_{n\in\mathscr{A}}\Big(\sum_{\substack{d|n\\ P^+(d)\le z}}\lambda_d\Big)^2\\ &= \sum_{P^+(d_1),P^+(d_2)\le z}\lambda_{d_1}\lambda_{d_2}\cdot\#\{n\in\mathscr{A} : [d_1,d_2]|n\},\end{aligned}$$

which is a quadratic form in the variables λ_d. Provided d_1 and d_2 are not too large, say at most R, one can hope to get a reasonable estimate for the coefficients $\#\{n\in\mathscr{A} : [d_1,d_2]|n\}$ of this quadratic form. The best upper bound stemming from this method then comes from minimizing the quadratic form over all choices of $\lambda_d\in\mathbb{R}$ with $\lambda_1 = 1$ and $\lambda_d = 0$ for $d > R$.

For typical sets $\mathscr{A}$ that arise in arithmetic problems, one finds that the optimal choice for the λ_d takes the form

$$\lambda_d \approx \mu(d)\cdot\left(\frac{\log(R/d)}{\log R}\right)^A \qquad (d\le R),$$

where A is some positive constant. We note that the weights λ_d decay to 0, and the larger the value of A, the higher the level of smoothness at the truncation point R. In the optimal choice, the exponent A is taken to be κ, the *dimension* of the sieve problem. However, for a given dimension κ, it is known [20, pg. 154] that any exponent $A>\kappa-1/2$ yields weights λ_d whose dominant contribution comes from numbers that have very few prime factors smaller than z, whereas this fails to be true for smaller A. See [8, ch. 10] for further discussion.

More generally, one can consider the smoothed sieve weight

$$M_f(n;R) := \sum_{d|n}\mu(d)f\left(\frac{\log d}{\log R}\right),$$

where $f:\mathbb{R}\to\mathbb{R}$ is a function supported on $(-\infty,1]$, which corresponds to taking $\lambda_d = \mu(d)f(\log d/\log R)$ for $d\le R$. In Selberg sieve arguments one typically chooses f to be a polynomial in [0, 1], perhaps of high degree. Such an example is offered by the 'GPY sieve' of Goldston-Pintz-Yıldırım [9, 24]. In more recent developments on gaps between primes by the third author [15] and Tao [21] one works with general smooth functions f.

The main motivation of this paper is to understand the exact role of the smoothing in the structure of the Selberg sieve weights. To this end, we consider their moments

$$\sum_{n\leq x} M_f(n;R)^k$$

as a tool of gaining additional insight on the distribution of the values of $M_f(n;R)$. On the practical side, higher moments naturally appear when applying Hölder's inequality, so it would be useful to know their behavior (1).

From the discussion above, in the case $f(x)=\max(1-x,0)^A$ and $k=2$, we have seen that if A is sufficiently large, then $M_f(n;R)^2$ 'behaves like a sieve weight' in the sense that the sum $\sum_{n<x} M_f(n;R)^2$ is $O_f(x/\log R)$ and the main contribution to this comes from numbers with few prime factors less than R. If A is too small and so f is not smooth enough, however, then $M_f(n;R)$ exhibits a qualitatively different behavior; the sum is larger than $x/\log R$, and the main contribution is no longer from numbers with few prime factors $\leq R$.

How smooth should f be so that $M_f(n;R)^{2k}$ behaves like a sieve weight when k varies, that is to say the main contribution to the $2k$-th moment (2) of $M_f(n;R)$ comes from integers a that have very few prime factors $\leq R$? What happens in the extreme case where f is the discontinuous function $\mathbf{1}_{(-\infty,1]}$? These are the types of questions that we will study in this paper.

1.1. Some smoothing is necessary to behave like a sieve weight

In order to gain a first understanding of the importance of smoothing, let us consider the sharp cut-off function

$$f_0 := \mathbf{1}_{(-\infty,1]}.$$

If $n=2m$ with m odd, then we have that

$$M_{f_0}(n;R)=\sum_{\substack{d|n\\ d\leq R}}\mu(d)=\sum_{\substack{d|m\\ d\leq R}}\mu(d)+\sum_{\substack{d|m\\ 2d\leq R}}\mu(2d)=\sum_{\substack{d|m\\ R/2<d\leq R}}\mu(d)=M_{\widetilde{f_0}}(m;R), \tag{1.2}$$

where, with a slight abuse of notation, we have put

$$\widetilde{f_0} := \mathbf{1}_{(1-\log 2/\log R,1]}. \tag{1.3}$$

In particular, if m is square-free and has exactly one divisor $d\in(R/2,R]$, then $M_{f_0}(n;R)=\pm 1$. An easy generalization of a deep result of Ford [5, Theorem 4] implies that (3) the proportion

(1) For example, Lemma 3.5 in Pollack's paper [17] is an example of a case where a fourth moment occurs because of the use of Cauchy's inequality, and a similar issue is encountered in Friedlander's work [7] for the combinatorial sieve instead of the Selberg sieve.

(2) We are typically interested in how large sieve weights get. If we took odd powers there might be an irrelevant cancelation, so we focus on even moments.

(3) The key estimates in the proof of the lower bound of Theorem 4 in [5] are the second part of Lemma 4.1, Lemma 4.3 (the parameters are $z=R\ll R/2=y$), Lemma 4.5, Lemma 4.8 and Lemma 4.9. A key observation is that only square-free integers are considered in Lemma 4.8, so that a stronger version of the lower bound of Theorem 4 of [5] can be immediately deduced by the same proof, that counts square-free integers with exactly one divisor in $(R/2,R]$.