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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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DECAY OF WEAKLY CHARGED SOLUTIONS FOR THE SPHERICALLY SYMMETRIC MAXWELL-CHARGED-SCALAR-FIELD EQUATIONS ON A REISSNER-NORDSTRÖM EXTERIOR SPACE-TIME

BY Maxime VAN DE MOORTEL

Abstract. – We consider the Cauchy problem for the (non-linear) Maxwell-Charged-Scalar-Field equations with spherically symmetric initial data, on a sub-extremal Reissner-Nordström or Schwarzschild exterior space-time. We prove that the solutions are bounded and decay at an inverse polynomial rate towards time-like infinity and along the black hole event horizon, provided the charge of the Maxwell equation is sufficiently small.

This condition is in particular satisfied for small data in energy space that enjoy a sufficient decay towards the asymptotically flat end.

Some of the decay estimates we prove are arbitrarily close to the conjectured optimal rate in the limit where the charge tends to zero, according to the heuristics present in the physics literature.

Our result can also be interpreted as a first step towards the stability of Reissner-Nordström black holes for the gravity coupled Einstein-Maxwell-Charged-Scalar-Field model. This problem is closely connected to the understanding of strong cosmic censorship and charged gravitational collapse in this setting.

Résumé. – On considère le problème de Cauchy pour les équations (non-linéaires) de Maxwell couplées avec un champ scalaire chargé, avec données initiales à symétrie sphérique, sur l'espace-temps de Reissner-Nordstrom sous-extrémal ou Schwarzschild. Nous démontrons que les solutions sont bornées et dispersent à un taux polynomial-inverse en temps large, vers timelike infinity et sur l'horizon des évènements, pourvu que la charge dans les équations de Maxwell soit suffisamment petite.

Cette condition est en particulier satisfaite pour des données de petite amplitude qui décroissent suffisamment rapidement à grandes distances dans la direction asymptotiquement plate.

La plupart des estimées que nous prouvons sont arbitrairement proches des taux optimaux conjecturés à la limite où la charge tend vers zéro, selon les arguments heuristiques présents dans la littérature en physique.

Notre résultat peut également être interprété comme la première étape en direction de la stabilité des trous noirs de Reissner-Nordstrom pour le modèle d'Einstein-Maxwell couplés avec un champ scalaire chargé. Ce problème est intimement lié à la conjecture de censure cosmologique forte, et à l'effondrement gravitationnel de matière chargée.

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1. Introduction

The model. – In this paper, we study the asymptotic behavior of solutions to the Maxwell-Charged-Scalar-Field equations, sometimes referred to as massless Maxwell-Klein-Gordon, arising from spherically symmetric and asymptotically decaying initial data on a fixed sub-extremal Reissner-Nordström exterior space-time:

$$\nabla^{\mu} F_{\mu\nu} = i q_0 \frac{(\phi \overline{D_{\nu}\phi} - \overline{\phi} D_{\nu}\phi)}{2}, \ F = dA, \tag{1.1}$$

$$g^{\mu\nu} D_{\mu} D_{\nu}\phi = 0, \tag{1.2}$$

$$g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2[d\theta^2 + \sin(\theta)^2 d\varphi^2], \tag{1.3}$$

$$\Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2}, \tag{1.4}$$

where $q_0 \geq 0$ is a constant called the charge (1) of the scalar field ϕ, M, ρ are respectively the mass and the charge of the Reissner-Nordström black hole with $0 \leq |\rho| < M$, ∇_{μ} is the Levi-Civita connection and $D_{\mu} = \nabla_{\mu} + i q_0 A_{\mu}$ is the gauge derivative. Note that—due to the interaction between the Maxwell field F and the charged scalar field ϕ—this system of equations is non-linear when $q_0 \neq 0$, the case of interest for this paper. This is in contrast to the uncharged case $q_0 = 0$, where (1.2) is then the linear wave equation.

Main results. – Since global regularity is known for this system (2), we focus on the asymptotic behavior.

The case of a charged scalar field on a black hole space-time that we consider is considerably different from the analogous problem on Minkowski space-time. While on the flat space-time, the charge of the Maxwell equation tends to 0 towards time-like infinity, this is not expected to be the case on black hole space-times. This fact constitutes a major difference and renders the proof of decay harder, already in the spherically symmetric case and when the charge is small.

We show that if the charge in the Maxwell equation and the scalar field energies are initially smaller than a constant depending on the black hole parameters M and ρ, then

1. the charge in the Maxwell equation is bounded and small on the Reissner-Nordström exterior space-time.
2. Boundedness of the scalar field energy holds.
3. A local integrated energy decay estimate holds for the scalar field.

If now we relax the smallness hypothesis on the charge, requiring only that the initial charge is smaller than a numerical constant (3) and assume 2 and 3, then

(1) This charge q_0 is also the coupling constant between the Maxwell and the scalar field equations. This is not to be confused with the charge of the Maxwell equation or the parameter ρ. For a precise definition of all "charges," cf. Section 2.3.

(2) It follows essentially from the global regularity of Yang-Mills equations on globally hyperbolic (3+1) Lorentzian manifolds, established in [8]. In spherical symmetry, this can also be deduced from the methods of [32].

(3) More precisely, the maximal value is $q_0|e_0| = \frac{1}{4}$ for the weakest claimed decay and $q_0|e_0| = 0.8267$ for the improved one, where e_0 is the initial asymptotic charge, cf. Section 2.3.

4. the energy [(4)] of the scalar field decays at an inverse polynomial rate, depending on the charge.
5. The scalar field enjoys point-wise decay estimates at an inverse polynomial rate consistent with 4.

These results are stated in a simplified version in Theorem 1.1 and later in a more precise way in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

The decay rate of the energy—and of some point-wise estimates that we derive—has been conjectured to be optimal in [27], in the limit when the asymptotic charge tends to zero, cf. Section 1.2.1. Other point-wise estimates, notably along the black hole event horizon, are however not sharp in that sense.

In the case of an uncharged scalar field $q_0 = 0$ (namely the wave equation on sub-extremal black holes), it is well-known that the long term asymptotics are governed by the so-called Price's law, first put forth heuristically by Price in [47] and later proved in [2, 1, 13, 21, 42, 51]. Generic solutions then decay in time at universal inverse polynomial decay rate, in the sense that this rate does not depend on physical parameters or on the initial data. In contrast, the optimal decay rate for our charged case $q_0 \neq 0$ is conjectured to be slower and depending on the charge in the Maxwell equation, itself determined by the data.

Roughly, this can be explained by the fact that in the uncharged case $q_0 = 0$, the equation effectively looks like the wave equation on Minkowski space in the presence of a potential decaying like r^{-3}, see Section 1.2.1.

This decay of the potential-like term is somehow more "forgiving" than in the charged case $q_0 \neq 0$. In the latter case, the equation becomes similar to the wave equation in the presence of a potential decaying like r^{-2}. As explained beautifully in pages 7 and 8 of [35], while the system is sub-critical with respect to the conserved energy in dimensions (3+1), the new term coming from the charge induces a form of criticality with respect to decay at space-like infinity, i.e., a criticality with respect to r weights.

One important consequence—in the black hole case—is that the sharp decay rate is expected to depend on the charge, cf. Section 1.2.1. Interestingly in our paper, in order to deal with this criticality, we need to use the full non-linear structure of the system. Also, the criticality with respect to decay implies the absence of any "extra convergence factor" that facilitates the proof of bounds on long time intervals, in the language of [35]. This is in contrast to the uncharged case and requires sharpness in the estimates, as explained in [35].

Motivation. – Our result can be viewed as a first step towards the understanding of the analogous Einstein-Maxwell-Charged-Scalar-Field model, where the Maxwell and Scalar Field equations are now coupled with gravity, cf. Equations (1.18), (1.19), (1.20), (1.21), (1.22) when $m^2 = 0$. In this setting, the asymptotic behavior of the scalar field is important, in particular because it determines the black hole interior structure, cf. Section 1.2.2. This is also closely related to the Strong Cosmic Censorship Conjecture, cf. Section 1.2.2.

Before discussing the relevance of our result for the interior of black holes, let us mention that the Einstein-Maxwell-Charged-Scalar-Field system possesses a number of new features

[(4)] By this, we mean all the energies transverse or parallel to the event horizon or null infinity, or L^2 flux on any constant r curve.